① $F(a)=\int_{-\infty}^{0} \underbrace{e^{x}\,\frac{\sinh(ax)}{x}}_{f(a,x)}\,dx.$ [8b]

(i) $x\to 0$: $\sinh ax \sim x$; $\Rightarrow$ $f(a,x)$ omezená v $(-\delta,0)$

integrál konečný pro $\forall a$ [1]

$x\to -\infty$: $\underbrace{\frac{1}{2x}}_{\text{nehraje roli}}\left(e^{x(1+a)}-e^{x(1-a)}\right).$

pomocný výpočet: $\int_{-\infty}^{0} e^{bx}\,dx$ ... je konečný a roven $\frac{1}{b}$ $\Leftrightarrow$ $b>0$.

tedy: $1+a>0$ & $1-a>0$: $a\in(-1,1)$. [1]

(ii) $f(a,\cdot)$ měřitelné: spojitost

$\frac{\partial f}{\partial a}(a,x) = e^{x}\cosh(ax)$ ... $\exists$ konečné $\forall x, \forall a$. [1]

majoranta: uvažuji $a\in I=(-\eta,\eta)$; $\eta\in(0,1)$ pevné.

$$\left|\frac{\partial f}{\partial a}\right| = e^{x}\cosh(ax) = e^{x}\cdot\frac{1}{2}\left(e^{ax}+e^{-ax}\right) \le \frac{1}{2}e^{x}\left(e^{-\eta x}\cdot 2\right)$$
$$= e^{x(1-\eta)} \in L^{1}(-\infty,0).$$ [2]

$f(0,x)=0\in L^{1}(-\infty,0)$. [1]

$$\Rightarrow F'(a)=\int_{-\infty}^{0} e^{x}\cosh ax = \frac{1}{2}\int_{-\infty}^{0} e^{x(1+a)}+e^{x(1-a)} = \frac{1}{2}\left(\frac{1}{1+a}+\frac{1}{1-a}\right)$$

iii) $F(a)=\frac{1}{2}\ln\frac{1+a}{1-a}+C$;

$F(0)=0 \Rightarrow C=0$. [2]

(2) $x^2+y^2=a^2$

$0\le z\le x+y$

$$\varphi:\begin{cases} x = a\cdot\cos u \\ y = a\cdot\sin u \\ z = v; \end{cases} \qquad \Omega:\begin{cases} u\in\left(-\frac{\pi}{4}, \frac{3\pi}{4}\right) \\ v\in\left(0, a(\cos u+\sin u)\right) \end{cases}$$

(i)

$0\le z: \ v\ge 0; \qquad x+y\ge 0: \ u\in\left(-\frac{\pi}{4}, \frac{3\pi}{4}\right)$ [2]

$$\partial_u\varphi = (-a\sin u, a\cos u, 0)$$
$$\partial_v\varphi = (0, 0, 1)$$
$$\times = (a\cos u, a\sin u, 0)$$

$\|\partial_u\varphi\times\partial_v\varphi\| = a \quad ; \quad f\circ\varphi = a(\cos u+\sin u)$ [1]

(ii)
$$\int_P f\,dS = \int_\Omega a^2(\cos u+\sin u)\,du\,dv = \int_{-\pi/4}^{\frac{3\pi}{4}}\left(\int_0^{a(\cos u+\sin u)} a^2(\cos u+\sin u)\,dv\right)du$$

$$= a^3\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}(\cos u+\sin u)^2\,du$$

$$= a^3\int_{-\frac{\pi}{4}}^{\frac{5\pi}{4}} \underbrace{\cos^2 u+\sin^2 u}_{1}+\underbrace{2\cos u\sin u}_{\sin 2u}\,du$$

$$= \pi a^3 + \underbrace{\int_{-\frac{\pi}{4}}^{\frac{5\pi}{4}}\sin 2u\,du}$$

$$= \left[-\frac{\cos 2u}{2}\right]_{-\frac{\pi}{4}}^{\frac{5\pi}{4}} = 0.$$ [2]

(iii) $\quad \omega = x^4 + y^4 ; \qquad \phi : \begin{array}{l} x = r (\cos u)^{\frac{1}{2}} \\ y = r (\sin u)^{\frac{1}{2}} \end{array}$

$$\phi^*(\omega) = r^4 \cos^2 u + r^4 \sin^2 u = r^4$$

[1]

$$d\phi^*(\omega) = 4 r^3 \, dr$$

$$d\omega = 4 \left(x^3 dx + y^3 dy \right) ; \qquad \begin{array}{l} dx = (\cos u)^{\frac{1}{2}} dr - r \frac{1}{2} (\cos u)^{-\frac{1}{2}} \sin u \, du \\ dy = (\sin u)^{\frac{1}{2}} dr + r \frac{1}{2} (\sin u)^{-\frac{1}{2}} \cos u \, du \end{array}$$

$$\phi^*(d\omega) = 4 \Big(r^3 (\cos u)^{3/2} \cdot \left(\cos^{\frac{1}{2}} dr - \cancel{r \tfrac{1}{2} (\cos u)^{-\frac{1}{2}} \sin u \, du} \right)$$

$$+ r^3 \sin^{3/2} \cdot \left(\sin^{\frac{1}{2}} dr + \cancel{r \tfrac{1}{2} (\sin u)^{-\frac{1}{2}} \cos u \, du} \right) \Big)$$

$$= 4 \left(r^3 \cos^2 u + r^3 \sin^2 u \right) dr = 4 r^3 \, dr \qquad [2]$$

③ $\phi(y) = \int_0^{\frac{\pi}{4}} \frac{(y')^2}{\cos x} + \frac{y}{(\cos x)^2}\, dx$; $y(0)=0$, $y(\frac{\pi}{4}) = \frac{1}{2}$ [8B]

$f = \frac{r^2}{\cos x} + \frac{y}{\cos^2 x}$; $f_r = \frac{2r}{\cos x}$, $f_y = \frac{1}{\cos^2 x}$;

E.L.: $-\left(\frac{2y'}{\cos x}\right)' + \frac{1}{\cos^2 x} = 0$ $\int$ [2]

$-\frac{2y'}{\cos x} + \operatorname{tg} x = c$;

$-2y' + \sin x = c \cdot \cos x$ $\int$

$-2y - \cos x = c \sin x + d$

rj: $y = -\frac{1}{2}(\cos x + c\sin x + d)$ [2]

$x=0$: $1 + d = 0$; $d = -1$

$y = \frac{\pi}{4}$: $-\frac{1}{2}\left(\frac{1}{\sqrt{2}} + c\frac{1}{\sqrt{2}} - 1\right) = \frac{1}{2}$;

$c = -1$. $y = \frac{1}{2}(\sin x - \cos x + 1)$ [1]

Jacobi: $f_{rr} = \frac{2}{\cos x} > 0$ pro $x \in [0, \frac{\pi}{4}] \Rightarrow$ podezření: minimum (1)

$f_{yr} = f_{yy} = 0$; $P(t) = \frac{2}{\cos x}$; $Q(t) = 0$

J.r. $-\left(\frac{2u'}{\cos x}\right)' = 0$

$\frac{u'}{\cos x} = c$

$u' = c\cos x$

$\boxed{u = d - c\sin x}$

$u(0) = 0$: $d = 0$

$u \not\equiv 0$: $c \neq 0$; rj $u = -c\sin x$

– nemá 0 v $(0, \frac{\pi}{4}]$

$\Rightarrow y = \frac{1}{2}(\sin x - \cos x + 1)$

je lokální minimum. [2]

[8b]

(4) $f(t) = \begin{cases} 1/\pi ; & x \in (-\frac{\pi}{2}, \frac{\pi}{2}) \\ 0 ; & x \in (-\pi, -\frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi) \end{cases}$

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sudá: $b_k = 0 \quad \forall k \geq 1$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\,dx = \frac{1}{\pi} \cdot 1 = \frac{1}{\pi}$$ [1]

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\,dx = \frac{2}{\pi^2} \int_0^{\pi/2} \cos kx = \frac{2}{\pi^2} \left[\frac{\sin kx}{k} \right]_0^{\frac{\pi}{2}} = \frac{2}{\pi^2 k} \sin \frac{k\pi}{2}$$

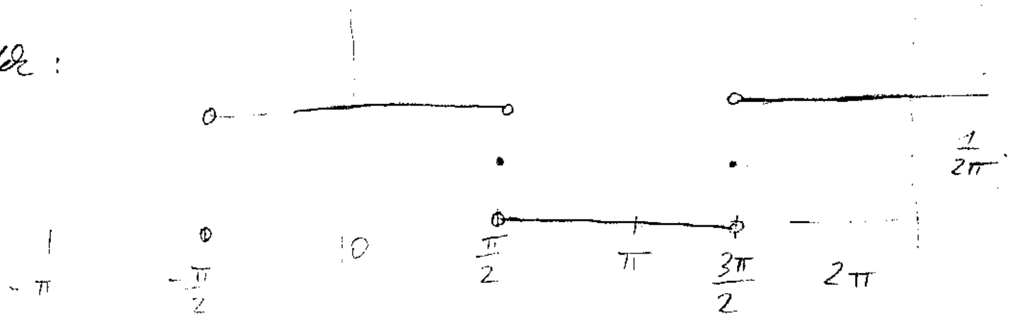
$$= \begin{cases} 0 ; & k = 2l \\ (-1)^l ; & k = 2l+1 \end{cases}$$

$$a_{2l+1} = \frac{2}{\pi^2 (2l+1)} \cdot \underbrace{\sin\left(\frac{\pi}{2} + l\pi\right)}_{(-1)^l}.$$ [2]

$$\mathcal{F}_f(t) = \frac{1}{2\pi} + \sum_{l=0}^{\infty} \frac{2}{\pi^2} \cdot \frac{(-1)^l}{2l+1} \cos(2l+1)x ;$$ [1]

$f(x)$ po částech C^1 : $\Rightarrow \mathcal{F}_f(t) = \begin{cases} f(t) \ldots & \text{bod spojitosti} \\ \frac{1}{2}\{f(t+) + f(t-)\} \ldots & \text{jinde} \end{cases}$

náčrtek:



[1]

$$x = 0 : \quad \frac{1}{\pi} = \frac{1}{2\pi} + \frac{2}{\pi^2}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

$$\Rightarrow \sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1} = \frac{\pi}{4}$$

Parseval: $\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2)$;

(LS): $\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\pi^2} = \frac{1}{\pi}\cdot\pi\cdot\frac{1}{\pi^2} = \frac{1}{\pi^2}$;

(PS): $\frac{1}{2\pi^2} + \sum_{l=0}^{\infty} \frac{4}{\pi^4(2l+1)^2}$;

[2]

řada: $\sum_{l=0}^{\infty} \frac{1}{(2l+1)^2} = \frac{\pi^2}{8}$

[1] .. za sečtení (známé) číselné řady.