and hence by (2.41), equating the two limits

$$\mathcal{L}(\Delta_a(t)) = e^{-as}.$$

This is just the expression obtained in (2.37).

The foregoing illustrates that the mathematical modeling of a sudden impulse is achieved rigorously by the treatment given in terms of the Riemann–Stieltjes integral.

Hereafter, for the sake of convenience, we will abuse the notation further and simply write

$$\mathcal{L}(\delta_a) = e^{-as}.$$

Example 2.30. A pellet of mass m is fired from a gun at time $t = 0$ with a muzzle velocity v_0. If the pellet is fired into a viscous gas, the equation of motion can be expressed as

$$m\,\frac{d^2x}{dt^2} + k\,\frac{dx}{dt} = m\,v_o\,\delta(t), \qquad x(0) = 0,\;\; x'(0) = 0,$$

where $x(t)$ is the displacement at time $t \geq 0$, and $k > 0$ is a constant. Here, $x'(0) = 0$ corresponds to the fact that the pellet is initially at rest for $t < 0$.

Taking the transform of both sides of the equation, we have

$$m\,s^2\,\mathcal{L}(x) + ks\,\mathcal{L}(x) = m\,v_0\,\mathcal{L}(\delta) = m\,v_0,$$

$$\mathcal{L}(x) = \frac{m\,v_0}{m\,s^2 + ks} = \frac{v_0}{s(s + k/m)}.$$

Writing

$$\frac{v_0}{s(s + k/m)} = \frac{A}{s} + \frac{B}{s + k/m},$$

we find that

$$A = \frac{m\,v_0}{k}, \qquad B = -\frac{m\,v_0}{k},$$

and

$$\mathcal{L}(x) = \frac{m\,v_0/k}{s} - \frac{m\,v_0/k}{s + k/m}.$$

The solution given by the inverse transform is

$$x(t) = \frac{m\,v_0}{k}\,(1 - e^{-\frac{k}{m}t})$$

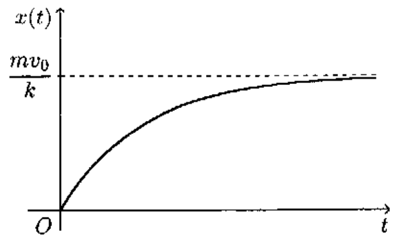

FIGURE 2.12

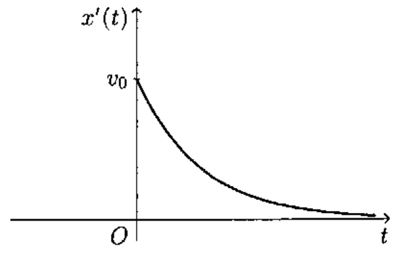

FIGURE 2.13

(Figure 2.12). Computing the velocity,

$$x'(t) = v_0\,e^{-\frac{k}{m}t},$$

and $\lim_{t\to 0^+} x'(t) = v_0$, whereas $\lim_{t\to 0^-} x'(t) = 0$, indicating the instantaneous jump in velocity at $t = 0$, from a rest state to the value v_0 (Figure 2.13).

Another formulation of this problem would be

$$m\,\frac{d^2x}{dt^2} + k\,\frac{dx}{dt} = 0, \qquad x(0) = 0,\;\; x'(0) = v_0.$$

Solving this version yields the same results as above.

Example 2.31. Suppose that at time $t = 0$ an impulse of $1V$ is applied to an RCL circuit (Figure 2.6), and for $t < 0$, $I(t) = 0$ and the charge on the capacitor is zero. This can be modeled by the equation

$$L\,\frac{dI}{dt} + RI + \frac{1}{C}\int_0^t I(\tau)\,d\tau = \delta(t),$$

where L, R, and C are positive constants, and

(i) $\dfrac{L}{C} > \dfrac{R^2}{4}$, **(ii)** $\dfrac{L}{C} < \dfrac{R^2}{4}$.

Applying the Laplace transform gives

$$Ls\,\mathcal{L}(I) + R\,\mathcal{L}(I) + \frac{1}{Cs}\,\mathcal{L}(I) = 1,$$

that is,

$$\begin{aligned}\mathcal{L}(I) &= \frac{s}{Ls^2 + Rs + 1/C} \\ &= \frac{s}{L[(s + R/2L)^2 + (1/LC - R^2/4L^2)]}.\end{aligned}$$

Setting $a = R/2L$, $b^2 = 1/LC - R^2/4L^2 > 0$, assuming (i), then,

$$\begin{aligned}L\,\mathcal{L}(I) &= \frac{s}{(s+a)^2 + b^2} \\ &= \frac{s+a}{(s+a)^2 + b^2} - \frac{a}{(s+a)^2 + b^2},\end{aligned} \qquad (2.42)$$

and so

$$I(t) = \frac{e^{-at}}{L}\left(\cos bt - \frac{a}{b}\,\sin bt\right).$$

Assuming (ii), (2.42) becomes

$$L\mathcal{L}(I) = \frac{s}{(s+a)^2 - b^2} = \frac{s+a}{(s+a)^2 - b^2} - \frac{a}{(s+a)^2 - b^2}$$

with $a = R/2L$, $b^2 = R^2/4L^2 - 1/LC > 0$. Consequently,

$$I(t) = \frac{e^{-at}}{L}\left(\cosh bt - \frac{a}{b}\sinh bt\right).$$

A Mechanical System. We consider a mass m suspended on a spring that is rigidly supported from one end (Figure 2.14). The rest position is denoted by $x = 0$, downward displacement is represented by $x > 0$, and upward displacement is shown by $x < 0$.

To analyze this situation let

i. $k > 0$ be the spring constant from Hooke's law,
ii. $a(dx/dt)$ be the damping force due to the medium (e.g., air), where $a > 0$, that is, the damping force is proportional to the velocity,
iii. $F(t)$ represents all external impressed forces on m.

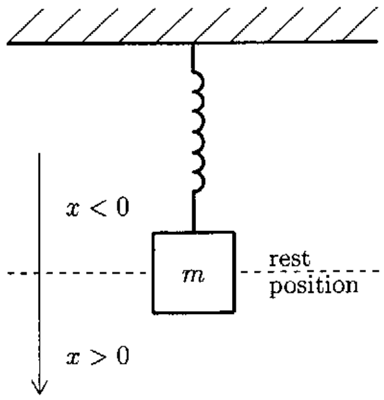

FIGURE 2.14

Newton's second law states that the sum of the forces acting on m equals $m\,d^2x/dt^2$, that is,

$$m\,\frac{d^2x}{dt^2} = -kx - a\,\frac{dx}{dt} + F(t),$$

or

$$m\,\frac{d^2x}{dt^2} + a\,\frac{dx}{dt} + kx = F(t). \qquad (2.43)$$

This equation is called the *equation of motion*.

Remark 2.32. If $a = 0$, the motion is called *undamped*. If $a \neq 0$, the motion is called *damped*. If $F(t) \equiv 0$ (i.e., no impressed forces), the motion is called *free*; otherwise it is *forced*.

We can write (2.43) with $F(t) \equiv 0$ as

$$\frac{d^2x}{dt^2} + \frac{a}{m}\,\frac{dx}{dt} + \frac{k}{m}\,x = 0.$$

Setting $a/m = 2b$, $k/m = \lambda^2$, we obtain

$$\frac{d^2x}{dt^2} + 2b\,\frac{dx}{dt} + \lambda^2 x = 0. \qquad (2.44)$$

The characteristic equation is

$$r^2 + 2br + \lambda^2 = 0,$$

with roots

$$r = -b \pm \sqrt{b^2 - \lambda^2}.$$

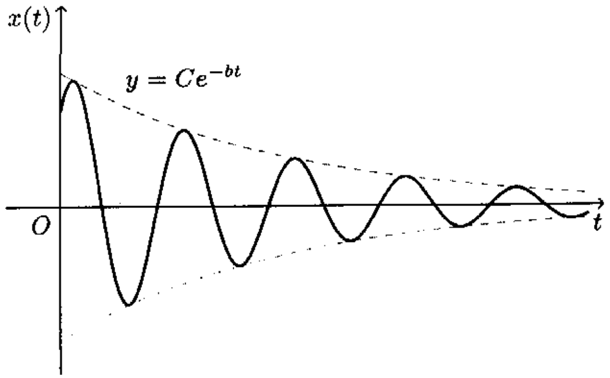

FIGURE 2.15

The resulting behavior of the system depends on the relation between b and λ. One interesting case is when $0 < b < \lambda$, where we obtain

$$x(t) = e^{-bt}(c_1 \sin\sqrt{\lambda^2 - b^2}t + c_2 \cos\sqrt{\lambda^2 - b^2}t),$$

which after some algebraic manipulations (setting $c = \sqrt{c_1^2 + c_2^2}$, $\cos\varphi = c_2/c$) becomes

$$x(t) = c\,e^{-bt}\cos(\sqrt{\lambda^2 - b^2}t - \varphi).$$

This represents the behavior of damped oscillation (Figure 2.15).

Let us apply a unit impulse force to the above situation.

Example 2.33. For $0 < b < \lambda$, suppose that

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \lambda^2 x = \delta(t), \qquad x(0) = 0, \; x'(0) = 0,$$

which models the response of the mechanical system to a unit impulse.

Therefore,

$$\mathcal{L}(x'') + 2b\,\mathcal{L}(x') + \lambda^2\mathcal{L}(x) = \mathcal{L}(\delta) = 1,$$

so that

$$\mathcal{L}(x) = \frac{1}{s^2 + 2bs + \lambda^2} = \frac{1}{(s+b)^2 + (\lambda^2 - b^2)},$$

and

$$x(t) = \frac{1}{\sqrt{\lambda^2 - b^2}}\,e^{-bt}\sin(\sqrt{\lambda^2 - b^2}t),$$

which again is a case of damped oscillation.

Exercises 2.5

1. Solve

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 2y = \delta(t), \qquad y(0) = y'(0) = 0.$$

2. The response of a spring with no damping ($a = 0$) to a unit impulse at $t = 0$ is given by

$$m\frac{d^2x}{dt^2} + kx = \delta(t), \qquad x(0) = 0, \; x'(0) = 0.$$

Determine $x(t)$.

3. Suppose that the current in an RL circuit satisifies

$$L\frac{dI}{dt} + RI = E(t),$$

where L, and R are constants, and $E(t)$ is the impressed voltage. Find the response to a unit impulse at $t = 0$, assuming $E(t) = 0$ for $t \leq 0$.

4. Solve

$$m\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = \delta(t),$$

for $m = 1$, $a = 2$, $k = 1$, $x(0) = x'(0) = 0$.

5. Show that if f satisfies the conditions of the derivative theorem (2.7), then

$$\mathcal{L}^{-1}(sF(s)) = f'(t) + f(0)\,\delta(t).$$

6. Show that

$$\mathcal{L}^{-1}\left(\frac{s-a}{s+a}\right) = \delta(t) - 2ae^{-at}.$$

7. A certain function $U(x)$ satisfies

$$a^2U'' - b^2U = -\frac{1}{2}\delta, \qquad x > 0,$$

where a and b are positive constants. If $U(x) \to 0$ as $x \to \infty$, and $U(-x) = U(x)$, show that

$$U(x) = \frac{1}{2ab}e^{-\frac{b}{a}|x|}.$$

[Hint: Take $U(0) = c$, $U'(0) = 0$, where c is to be determined.]

2.6 Asymptotic Values

Two properties of the Laplace transform are sometimes useful in determining limiting values of a function $f(t)$ as $t \to 0$ or as $t \to \infty$, even though the function is not known explicitly. This is achieved by examining the behavior of $\mathcal{L}(f(t))$.

Theorem 2.34 (Initial-Value Theorem). *Suppose that f, f' satisfy the conditions as in the derivative theorem* (2.7), *and $F(s) = \mathcal{L}(f(t))$. Then*

$$f(0^+) = \lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s) \qquad (s \text{ real}).$$

Proof. By the general property of all Laplace transforms (of functions), we know that $\mathcal{L}(f'(t)) = G(s) \to 0$ as $s \to \infty$ (Theorem 1.20). By the derivative theorem,

$$G(s) = sF(s) - f(0^+), \qquad s > \alpha.$$

Taking the limit,

$$0 = \lim_{s\to\infty} G(s) = \lim_{s\to\infty} \left(sF(s) - f(0^+)\right).$$

Therefore,

$$f(0^+) = \lim_{s\to\infty} sF(s).$$

□

Example 2.35. If

$$\mathcal{L}(f(t)) = \frac{s+1}{(s-1)(s+2)},$$

then

$$f(0^+) = \lim_{s\to\infty} s\left(\frac{s+1}{(s-1)(s+2)}\right) = 1.$$

Theorem 2.36 (Terminal-Value Theorem). *Suppose that f satisfies the conditions of the derivative theorem* (2.7) *and furthermore that $\lim_{t\to\infty} f(t)$ exists. Then this limiting value is given by*

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) \qquad (s \text{ real}),$$

where $F(s) = \mathcal{L}(f(t))$.

Proof. First note that f has exponential order $\alpha = 0$ since it is bounded in view of the hypothesis. By the derivative theorem,

$$G(s) = \mathcal{L}(f'(t)) = sF(s) - f(0^+) \qquad (s > 0).$$

Taking the limit,

$$\lim_{s\to 0} G(s) = \lim_{s\to 0} sF(s) - f(0^+). \tag{2.45}$$

Furthermore,

$$\lim_{s\to 0} G(s) = \lim_{s\to 0} \int_0^\infty e^{-st} f'(t)\,dt$$
$$= \int_0^\infty f'(t)\,dt, \tag{2.46}$$

since in this particular instance the limit can be passed inside the integral (see Corollary A.4). The integral in (2.46) exists since it is nothing but

$$\int_0^\infty f'(t)\,dt = \lim_{\tau\to\infty} \int_0^\tau f'(t)\,dt$$
$$= \lim_{\tau\to\infty} [f(\tau) - f(0^+)]. \tag{2.47}$$

Equating (2.45), (2.46), and (2.47),

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

□

Example 2.37. Let $f(t) = \sin t$. Then

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{s}{s^2+1} = 0,$$