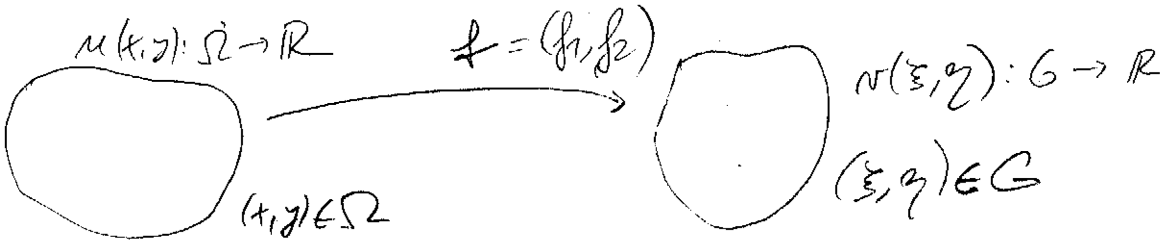
Konformní zobrazení. (v $\mathbb{R}^2 = \mathbb{C}$)



f .. je konformní: 1) vzájemně jednoznačné

2) holomorfní; tj. $\exists f'(z)$; $z = x+iy$

3) $f'(z) \neq 0$ ($\in$)

Pozn.: • holomorfní $\Rightarrow$ C.R. podmínky $\dfrac{\partial f_1}{\partial x} = \dfrac{\partial f_2}{\partial y}$

$$f'(z) = \frac{\partial f_1}{\partial x} + i\frac{f_2}{\partial x} \qquad \frac{\partial f_1}{\partial y} = -\frac{\partial f_2}{\partial x}$$

• f konf. $\Rightarrow f \in C^\infty$; f^{-1} konformní.

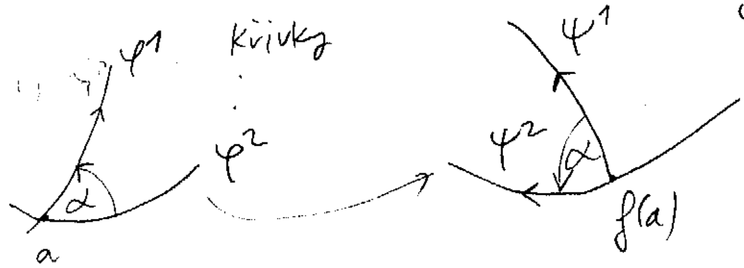
• $\Delta f_1 = \Delta f_2 = 0$ ($\in$ C.R.)

• $$Jf = \det(\nabla f) = \begin{vmatrix} \partial_x f_1, & \partial_y f_1 \\ \partial_x f_2, & \partial_y f_2 \end{vmatrix} = (\partial_x f_1)^2 + (\partial_x f_2)^2 = |f'(z)|^2 = h^2$$

C.R.: $\partial_x f_2 = -\partial_y f_1$, $\partial_y f_2 = \partial_x f_1$

h .. Lamého koeficient.

• f konf. $\Rightarrow$ f zachovává úhly. $\psi^i = f \circ \varphi^i$



· Motivace: $\Delta u = \rho$ N.S.C. u ... el. potenciál

$$u|_{\partial\Omega} = u_0 \qquad -\nabla u = E \text{ el. pole}$$

$\uparrow$ hranice Ω $\qquad \rho$ -- náboj

u_0 -- okr. podmínka

Věta 1: $f : \Omega \to G$ konf.; $v : G \to \mathbb{R}$, $u : \Omega \to \mathbb{R}$

$$u = v \circ f$$

$$u(x,y) = v(\underbrace{f_1(x,y)}_{\xi}, \underbrace{f_2(x,y)}_{\eta}).$$

1. $u, v \in C^2$: $\Delta_{xy} u = (\Delta_{\xi\eta} v) \circ f \cdot h^2$

2. $-\Delta v = \delta_a \Rightarrow -\Delta u = \delta_{f^{-1}(a)}$ $(v\ \mathcal{D}')$.

$u, v \in L^1_{loc}$

3. $E := -(\partial_x u + i\partial_y u)$ $\Rightarrow$ $E = (\tilde{E} \circ f) \cdot \overline{f'(\Omega)}$

$\tilde{E} := -(\partial_\xi v + i\partial_\eta v)$

$\Delta f_1 = \Delta f_2 = 0$

$= 0$

Dk. 1: $\partial_x u = \partial_\xi v \cdot \partial_x f_1 + \partial_\eta v \cdot \partial_x f_2 \quad / \ \partial_x$

Čihák s.121

$$\partial_{xx} u = (\partial_{\xi\xi} v \cdot \partial_x f_1 + \partial_{\xi\eta} v \cdot \partial_x f_2)\partial_x f_1 + \partial_\xi v \cdot \partial_{xx} f_1$$
$$+ (\partial_{\xi\eta} v \cdot \partial_x f_1 + \partial_{\eta\eta} v \cdot \partial_x f_2) \cdot \partial_x f_2 + \partial_\eta v \cdot \partial_{xx} f_2$$

$$\partial_{yy} u = (\partial_{\xi\xi} v \cdot \partial_y f_1 + \partial_{\xi\eta} v \cdot \partial_y f_2) \cdot \partial_y f_1 + \partial_\xi v \cdot \partial_{yy} f_1$$
$$+ (\partial_{\xi\eta} v \cdot \partial_y f_1 + \partial_{\eta\eta} v \cdot \partial_y f_2) \cdot \partial_y f_2 + \partial_\eta v \cdot \partial_{yy} f_2$$

$$\Delta u = \partial_{\xi\xi} v \left[(\partial_x f_1)^2 + \underset{\substack{\| \\ (\partial_x f_2)^2}}{(\partial_y f_1)^2} \right] + \partial_{\eta\eta} v \cdot \left[(\partial_x f_2)^2 + \underbrace{(\partial_y f_2)^2}_{=(\partial_x f_1)^2} \right] \cdot$$

$$\partial_{\xi\eta} v \cdot \left[\partial_x f_2 \, \partial_x f_1 + \partial_x f_1 \, \partial_x f_2 + \partial_y f_2 \, \partial_y f_1 + \partial_y f_1 \, \partial_y f_2 \right]$$

$$+ \; \partial_\xi v \cdot \Delta f_1 + \partial_\eta v \cdot \Delta f_2 .$$

Ad 2: $\langle -\Delta u, \varphi \rangle \underset{\text{def. derivace}}{=} \langle u, -\Delta \varphi \rangle = \int_\Omega u(x,y) \left(-\Delta \varphi(x,y) \right) dx\, dy$

lib. $\varphi \in \mathcal{D}(\Omega)$ def. derivace.

nechť: $\varphi = \psi \circ f$, $\psi \in \mathcal{D}(G)$.

$$= \int_\Omega v(f(x,y)) \cdot \Big(\underbrace{-\Delta(\psi \circ f)}_{(-\Delta\psi)\circ f \cdot h^2}(x,y) \Big) dx\, dy \qquad \psi = \varphi \circ f_{-1} \in \mathcal{D}(G)$$

... bod 1.

$$h^2 = |Jf|$$

$$= \int_\Omega (v \circ f) \cdot (-\Delta \psi) \circ f \cdot |Jf| \, dx\, dy$$ věta o subst.

$$= \int_G v \cdot (-\Delta \psi) \, d\xi\, d\eta = \langle -\Delta v, \psi \rangle$$

$$= \langle \delta_a, \psi \rangle = \psi(a) = \varphi(f_{-1}(a))$$

$$= \langle \delta_{f_{-1}(a)}, \varphi \rangle .$$

Př. 3.

$$E = -(\partial_x u + i\,\partial_y u)$$

$$= -\Big[\big(\partial_\xi v \cdot \partial_x f_1 + \partial_\eta v \cdot \partial_x f_2\big) + i\big(\partial_\xi v \cdot \underbrace{\partial_y f_1}_{-\partial_x f_2} + \partial_\eta v \cdot \underbrace{\partial_y f_2}_{\partial_x f_1}\big)\Big]$$

$$= -\Big[\partial_\xi v\,(\partial_x f_1 - i\,\partial_x f_2) + \partial_\eta v\,(\partial_x f_2 + i\,\partial_x f_1)\Big]$$

$$i\,\partial_\eta v\,(\partial_x f_1 - i\,\partial_x f_2)\Big]$$

$$= -\,(\partial_\xi v + i\,\partial_\eta v)(\partial_x f_1 - i\,\partial_x f_2)$$

$$= -\,\tilde{E} \cdot \overline{f'(z)}$$

Fakta: ① $v = \frac{1}{2\pi} \ln \frac{1}{r} = \frac{-1}{4\pi} \ln(\xi^2 + \eta^2)$ Čihák, s. 39

řeší $-\Delta v = \delta_0$ v $\mathcal{D}'(\mathbb{R}^2)$

② $u_0(x,y) = \dfrac{y}{\pi(x^2+y^2)}$ řeší $-\Delta u = 0$

$u\big|_{y=0+} = \delta_0(x)$.

$\Rightarrow$ $g(x): \mathbb{R} \to \mathbb{R}$ omezená fce s. 58; Př. 2.44.

$u(x,y) = g * u_0(\cdot, y)$ řeší

$$\boxed{\begin{array}{l} -\Delta u = 0 \\ u\big|_{y=0} = g \end{array}}$$