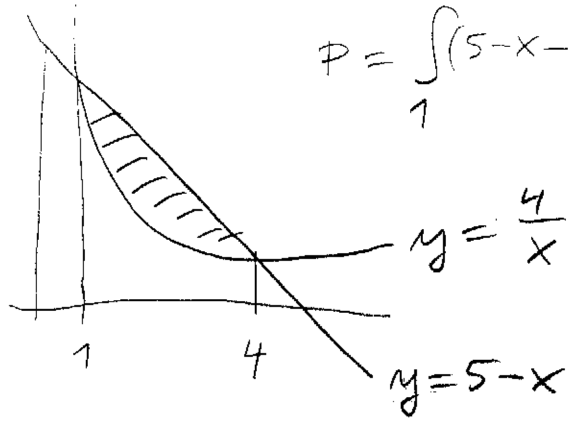
(1)

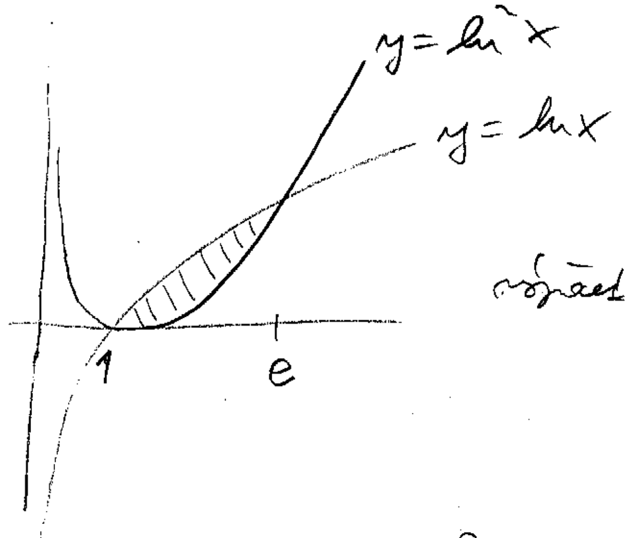


$$P = \int_1^4 \left(5 - x - \frac{4}{x}\right) dx = \left[5x - \frac{x^2}{2} - 4\ln|x|\right]_1^4$$

$$= \frac{15}{2} - 4\ln 4 .$$

m.x.

(2)



$$P = \int_1^e \ln x - \ln^2 x \, dx = 3 - e$$

výpočet: $\int 1 \cdot (\ln x - \ln^2 x) dx$

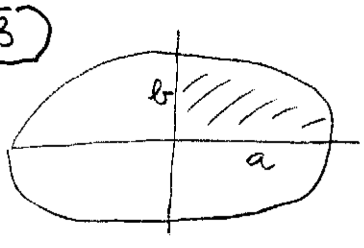
$u' = 1 \; ; \; v = \ln x - \ln^2 x$

$u = x \; ; \; v' = \frac{1}{x} - 2\frac{\ln x}{x}$

$$= x(\ln x - \ln^2 x) - \int 1 - 2\ln x \, dx \; ; \text{ etc} ;$$

celkem: $\int \ln x - \ln^2 x \, dx = -x\ln^2 x + 3x\ln x - 3x$ m.x.

(3)



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \; ;$$

$$|y| \le b\sqrt{1 - \frac{x^2}{a^2}} \; ; \quad x \in [-a, a].$$

$$\frac{1}{4}P = \int_0^a b\sqrt{1 - \left(\frac{x}{a}\right)^2} dx \;\left|\; \begin{matrix} x = a \cdot \sin t \\ dx = a\cos t \, dt \\ t \in (0, \frac{\pi}{2}) \end{matrix} \right\} = \int_0^{\frac{\pi}{2}} ab\cos^2 t \, dt$$

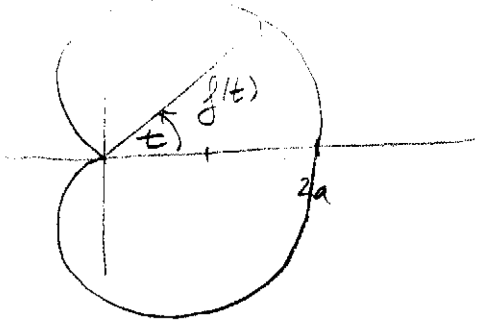
$$= ab \int_0^{\frac{\pi}{2}} \left(\frac{1 + \cos 2t}{2}\right) dt = ab\left[\frac{t}{2} + \frac{1}{4}\sin 2t\right]_0^{\frac{\pi}{2}} = \frac{\pi ab}{4}.$$

vzorečky: $\cos^2 y = \frac{1}{2}(1 + \cos 2y)$

$\sin^2 y = \frac{1}{2}(1 - \cos 2y)$

celkem: $P = \underline{\pi ab}$

(4) $f(t) = a(1+\cos t)$; $t \in [0, 2\pi]$.

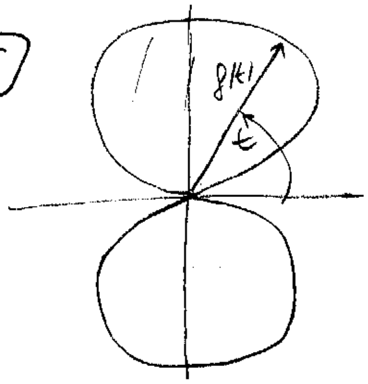


vzorec: $P = \frac{1}{2}\int_0^{2\pi} f^2(t)\,dt$

$$= \frac{a^2}{2}\int_0^{2\pi}(1+\cos t)^2\,dt = \frac{a^2}{2}\int_0^{2\pi} 1+2\cos t+\underbrace{\cos^2 t}_{\frac{1}{2}(1+\cos 2t)}\,dt$$

$$= \frac{a^2}{2}\int_0^{2\pi}\frac{3}{2}+2\cos t+\frac{1}{2}\cos 2t\,dt = \frac{3}{2}\pi a^2$$ m.x. w.k.

(5)



$$P = \frac{1}{2}\int_0^{2\pi}(4\sin^2 t)^2\,dt = 8\int_0^{2\pi}\sin^4 t\,dt;$$

$$\sin^4 t = (\sin^2 t)^2 = \left(\frac{1}{2}(1-\cos 2t)\right)^2$$

$$= \frac{1}{4}\left(1-2\cos 2t+\cos^2 2t\right)$$

$$= \frac{1}{4}\left(1-2\cos 2t+\frac{1}{2}(1+\cos 4t)\right);$$

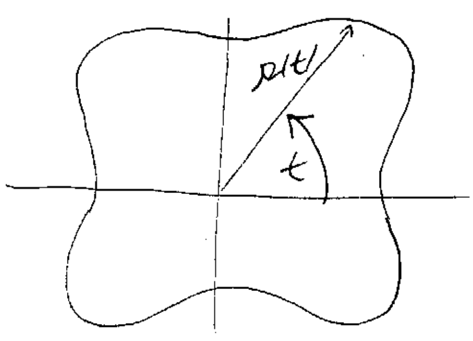
odsud lehce: $\int_0^{2\pi}\sin^4 t\,dt = \frac{3\pi}{4}$; $P = 6\pi$.

(6) $x^4+y^4 = x^2+y^2 \Rightarrow r^4(t)\left(\cos^4 t+\sin^4 t\right) = r^2(t)$

$x = r(t)\cos t$

$y = r(t)\sin t$

$$r(t) = \frac{1}{\sqrt{\cos^4 t+\sin^4 t}}$$



$$P = \frac{1}{2}\int_0^{2\pi} r^2(t) = \frac{1}{2}\int_0^{2\pi}\frac{dt}{\cos^4 t+\sin^4 t}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{dt}{\cos^4 t+\sin^4 t};$$

$\underline{6}$ (dokončení):
$$u = \operatorname{tg} t \; ; \quad t \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \leftrightarrow u \in (-\infty, \infty)$$
$$dt = \frac{du}{1+u^2} \; ; \quad \cos^2 t = \frac{1}{1+u^2} \; ; \quad \sin^2 t = \frac{u^2}{1+u^2}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\frac{1}{(1+u^2)^2} + \frac{u^4}{(1+u^2)^2}} \cdot \frac{du}{1+u^2} = \int_{-\infty}^{\infty} \frac{1+u^2}{1+u^4}\, du .$$

rozklad: $u^4+1 = (u^2+1)^2 - (\sqrt{2}u)^2 = (u^2+\sqrt{2}u+1)\cdot(u^2-\sqrt{2}u+1)$

$$\frac{u^2+1}{u^4+1} = \frac{Au+B}{u^2+\sqrt{2}u+1} + \frac{Cu+D}{u^2-\sqrt{2}u+1} \qquad \begin{array}{l} A = C = 0 \\ B = D = \frac{1}{2}. \end{array}$$

$$= \frac{1}{2}\left(\frac{1}{u^2+\sqrt{2}u+1} + \frac{1}{u^2-\sqrt{2}u+1}\right) ;$$

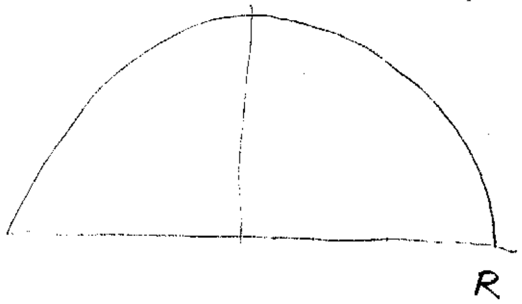
pomocný vzoreček: $(p^2-4q<0)$:
$$\int_{-\infty}^{\infty} \frac{du}{u^2+pu+q} = \int_{-\infty}^{\infty} \frac{du}{\left(u+\frac{p}{2}\right)^2 + q - \frac{p^2}{4}} \qquad \begin{array}{|l} u+\frac{p}{2} = s \\ q-\frac{p^2}{4} = c^2 \end{array}$$

$$= \int_{-\infty}^{\infty} \frac{ds}{s^2+c^2} = \frac{1}{c^2}\int_{-\infty}^{\infty} \frac{ds}{\left(\frac{s}{c}\right)^2+1} = \frac{\pi}{c} = \underline{\frac{\pi}{\sqrt{q-p^2/4}}}$$

odtud tedy: $P = \frac{1}{2}\cdot\pi\left(\frac{1}{\sqrt{1-\frac{1}{2}}} + \frac{1}{\sqrt{1-\frac{1}{2}}}\right) = \pi\sqrt{2}.$

m.x.

(7) koule: $y=\sqrt{R^2-x^2}$; $x\in[-R,R]$

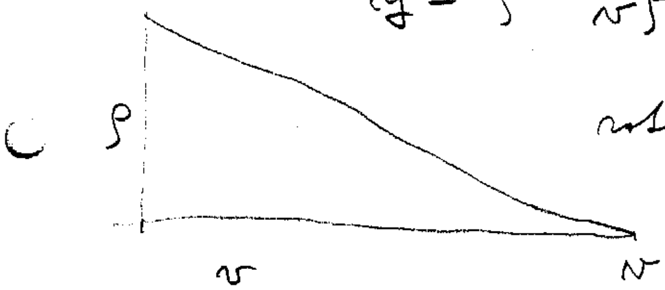


rotace dle osy x

$$V=\pi\int_a^b f^2(x)\,dx=\pi\int_{-R}^{R}R^2-x^2=\pi\frac{4}{3}R^3$$

kužel:

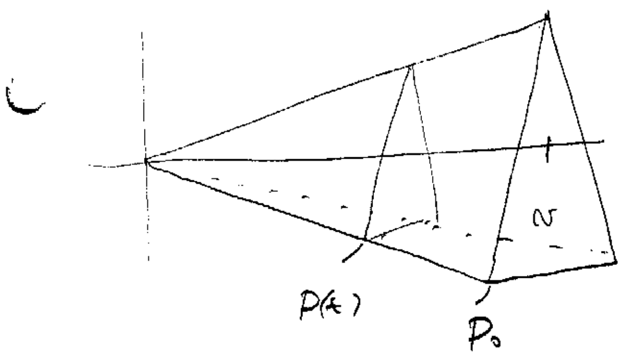
$$y=\rho-\frac{x}{v}\rho\ ;\quad x\in[0,v]$$



rotace dle osy x:

$$V=\pi\int_0^v\rho^2\left(1-\frac{x}{v}\right)^2dx=\dots=\pi\rho^2\frac{v}{3}.$$

jehlan



$$V=\int_0^v P(x)\,dx\ ;$$

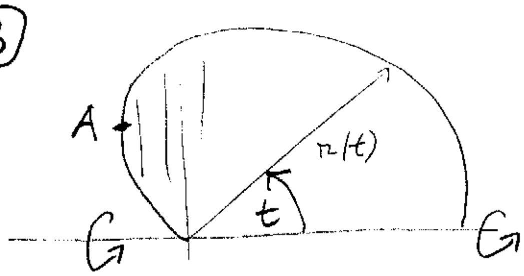
kde $P(x)$ ... plocha „řezu" ve výšce x.

řez je $\frac{x}{v}$ - krát menší, tedy

$$P(x)=\left(\frac{x}{v}\right)^2\cdot\underbrace{P_0}_{\uparrow\ \text{plocha podstavy}}$$

celkem: $V=\int_0^v\left(\frac{x}{v}\right)^2P_0\,dx=\frac{1}{3}vP_0.$

8



$x = r(t)\cos t$; $r(t) = a(1+\cos t)$

$y = r(t)\sin t$

$t \in [0, \pi]$.

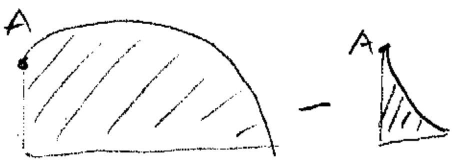
poznámka: $x = \varphi(t)$; φ prosté ; $\psi \geq 0$:
$y = \psi(t)$

$$V = \pi \int_a^b \psi^2 |\varphi'(t)|\, dt.$$

úvaha: $\varphi' < 0$; $t \in [0, \frac{2\pi}{3}]$; $t = \frac{2\pi}{3} \sim A$

$\varphi' > 0$; $t \in [\frac{2\pi}{3}, \pi]$ ... tuto část objemu „odčítáme"

$\varphi = -\sin t(2\cos t + 1)$

$V =$  .

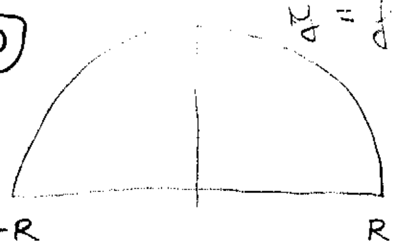
celkem: $V = \pi \int_0^\pi \psi^2(t)(-\varphi'(t))\, dt$

$$= \pi \int_0^\pi \left(a(1+\cos t)\sin t\right)^2 \left(-a(1+\cos t)\cos t\right)' dt$$

$$= \pi a^3 \int_0^\pi \cos^2 t(\cos t + 1)^2(2\cos t + 1)\sin t\, dt \quad \Bigg| \begin{array}{l} -y = \cos t \\ dy = \sin t\, dt \\ y \in (-1, 1) \end{array}$$

$$= \pi a^3 \int_{-1}^{1} y^2(1-y)^2(1-2y)\, dy = \dots = \frac{8}{3} \text{ m.x.}$$

(10)



$y = f(x)$; $\quad l = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$;

$$f(x) = \sqrt{R^2 - x^2}.$$

$$f'(x) = \frac{-x}{\sqrt{R^2-x^2}}.$$

$$o = 2\cdot\int_{-R}^{R}\sqrt{1+\frac{x^2}{R^2-x^2}}\,dx = 2R\int_{-R}^{R}\frac{dx}{\sqrt{R^2-x^2}} \quad \Bigg| \begin{array}{l} x = R\sin t \\ dx = R\cos t\,dt \\ t\in\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \end{array}$$

$$= 2R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{R\cos t}{\sqrt{R^2-R^2\sin^2 t}}\,dt = 2R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\underbrace{\frac{\cos t}{|\cos t|}}_{=1 \text{ на } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}dt = 2\pi R.$$

(11) $f(x) = \arcsin x + \sqrt{1-x^2}$

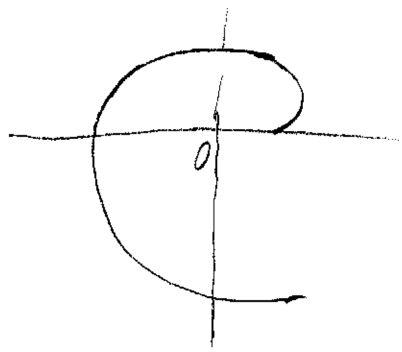
$$f'(x) = \frac{1-x}{\sqrt{1-x^2}}$$

$$l = \int_{-1}^{1}\sqrt{1+\frac{(1-x)^2}{1-x^2}}\,dx = \int_{-1}^{1}\sqrt{1+\frac{1-x}{1+x}}\,dx = \int_{-1}^{1}\sqrt{\frac{2}{1+x}}\,dx$$

$$= \left[\sqrt{2}\cdot 2\sqrt{1+x}\right]_{-1}^{1} = 4 \text{ m.x.}$$

(12)

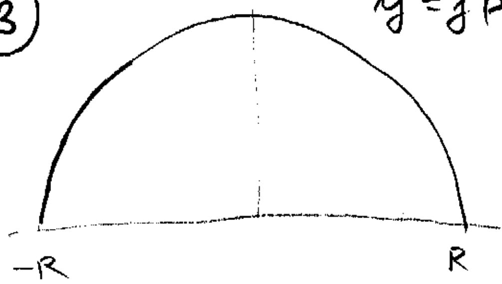
$$x = \varphi(t) = a(\cos t + t \sin t)$$
$$y = \psi(t) = a(\sin t - t\cos t)$$
$$t \in [0, 2\pi]$$



$$l = \int_0^{2\pi} \sqrt{(\varphi')^2 + (\psi')^2}\, dt \; ; \quad \varphi' = at\cos t\;; \quad \psi' = at\sin t\;;$$

$$= a\int_0^{2\pi} \sqrt{t^2\cos^2 t + t^2 \sin^2 t}\, dt = a\int_0^{2\pi} \underbrace{|t|}_{= t \text{ na } (0,2\pi)}\, dt = 2a\pi^2.$$

(13)



$$y = f(x)\;; \; x \in [a, b]$$

vzorec: $S = 2\pi \int_a^b |f(x)| \sqrt{1 + [f'(x)]^2}\, dx$

$$f(x) = \sqrt{R^2 - x^2}$$
$$f'(x) = \frac{-x}{\sqrt{R^2 - x^2}}$$

$$S = 2\pi \int_{-R}^{R} \sqrt{R^2 - x^2} \cdot \sqrt{1 + \frac{x^2}{R^2 - x^2}}\, dx$$

$$= 2\pi \int_{-R}^{R} \sqrt{R^2}\, dx = 4\pi R^2.$$

(14) $f(x) = x^3$; $\ell = \int_{-1}^{1} |x^3| \sqrt{1+9x^4}\,dx$

$f'(x) = 3x^2$

$= 2\int_0^1 x^3 \sqrt{1+9x^4}\,dx$ | $9x^4+1 = u$; $u \in [1,10]$

$36x^3 = du$

$= \frac{2}{36}\int_1^{10} \sqrt{u}\,du = \frac{2}{36}\cdot\left[\frac{2}{3}u^{3/2}\right]_1^{10} = \frac{1}{27}\left(10\sqrt{10}-1\right).$

m.x.