**1.1.** $f = x+y$

$M = \{x^2+y^2 = 1\}$

VZOREC: $\sin\alpha + \sin\beta = 2 \sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

↑ parametr... $x = \cos t$, $y = \sin t$; $t \in [0, 2\pi]$

$$\varphi(t) = f(\cos t, \sin t) = \cos t + \sin t = \sin\left(t+\tfrac{\pi}{2}\right) + \sin t$$
$$= 2\sin\left(t+\tfrac{\pi}{4}\right)\cdot\cos\tfrac{\pi}{4} = \sqrt{2}\sin\left(t+\tfrac{\pi}{4}\right).$$

max: $t = \frac{\pi}{4}$, $\rightarrow \left(\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right)$

min: $t = \frac{5\pi}{4}$ $\rightarrow \left(-\frac{1}{\sqrt2}, -\frac{1}{\sqrt2}\right)$.

**1.2.** $f = e^x$

$M = \{x^2+2y^2 = 1\}$.

$e^x$ max $\Leftrightarrow x$ max $\Leftrightarrow (x,y) = (1,0)$

min $\Leftrightarrow x$ min $\Leftrightarrow (x,y) = (-1,0)$.

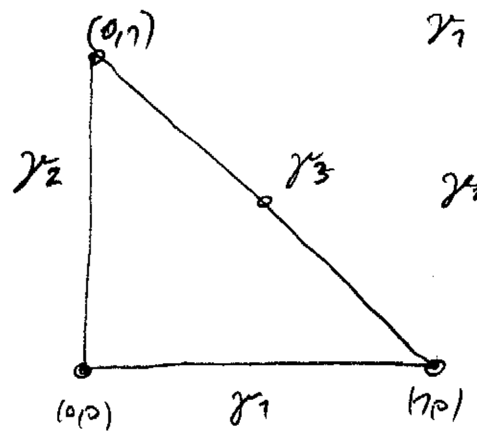
**1.3.** $f = x^2+y$

$M = \{x \geq 0\} \cap \{y \geq 0\} \cap \{x+y \leq 1\}$.

$M$ omezená: $x, y \in [0,1]$
uzavřená: průnik uzavřených
$\Big\}$ $\Rightarrow \exists$ glob. extrémy.

($\alpha$) uvnitř: $\nabla f = (2x, 1) = (0,0)$ $\emptyset$

($\beta$) hranice:




$\gamma_1$: $\varphi(t) = f(t,0)$; $t \in [0,1]$
$= t^2$ ... rostoucí (0,0) MIN, (1,0) MAX

$\gamma_2$: $\varphi(t) = f(0,t) = t$
... roste: (0,1) MAX

$\gamma_3$: $\varphi(t) = f(t, 1-t) = t^2+1-t$
$\varphi'(t) = 2t-1$ ... $t = \frac{1}{2}$:
$\left(\frac12, \frac12\right)$ $f = \frac34$

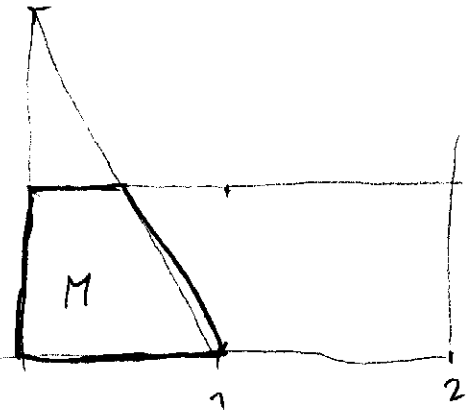
(0,0): glob. min

(1,0), (0,1): glob. max.

**1.4.** $f = x$

$M = \{0 \le x \le 2\} \cap \{0 \le y \le 1\} \cap \{2x + y \le 2\}$.

$y \le 2(1-x)$

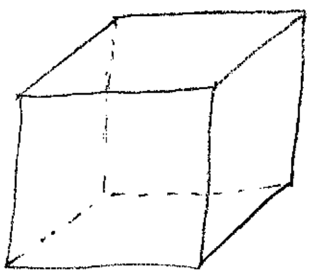



$x$ max: $(1,0)$

min: $(0,t)$; $t \in [0,1]$.

**1.5.** $f = (x+y)^2 + (x-y)^2 + z$

$M = [-1,1]^3$.



(α) vnitřek: $\nabla f = 0$ …

$2(x+y) + 2(x-y) = 0$

$2(x+y) - 2(x-y) = 0$

$\underline{1 = 0}$ $\emptyset$.

(β) hranice: $\frac{\partial f}{\partial z} = 1$ … roste vůči $z$:

$\Rightarrow$ max pro $z = 1$ :

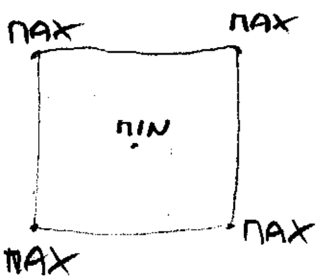
min pro $z = -1$;

---

stačí vyšetřit $F(x,y) = (x+y)^2 + (x-y)^2$

$= 2(x^2 + y^2)$

na čtverci $N = [-1,1] \times [-1,1]$.




Závěr pro $f$: $(\pm 1, \pm 1, 1)$ : max.

$(0, 0, -1)$ : min

**1.6.** $f = \frac{x}{a} + \frac{y}{b}$ ; $a, b > 0$

$M = \{x^2 + y^2 \leq 1\}$.

(α) vnitřek $\nabla f = \underset{\sim}{0}$ ; $\nabla f = \left(\frac{1}{a}, \frac{1}{b}\right) \dots \emptyset$

(β) hranice: $\varphi(t) = f(\cos t, \sin t) = \frac{1}{a}\cos t + \frac{1}{b}\sin t$

<u>trik:</u> $\left(\frac{1}{a}, \frac{1}{b}\right) = \rho(\cos\omega, \sin\omega)$; $\rho > 0$, $\omega \in (0, \frac{\pi}{2})$.

kde $\rho = \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$ ;

$\omega = \operatorname{arctg}\left(\frac{a}{b}\right)$.

$\varphi(t) = \rho \cos\omega \cos t + \sin\omega \sin t$

$= \rho \cos(\omega - t)$ ;

max: $t = \omega$ ... bod $(x, y) = \frac{1}{\rho}\left(\frac{1}{a}, \frac{1}{b}\right)$

min: $t = \omega + \pi$ $(x, y) = -\frac{1}{\rho}\left(\frac{1}{a}, \frac{1}{b}\right)$.

**1.7.** $f = x^2 + y^2 + z^2 + 2x + 4y - 6z$

$M = \mathbb{R}^3$.

$\nabla f = \underset{\sim}{0}$: $2x + 2 = 0$ $(x, y, z) = (-1, -2, 3)$

$2y + 4 = 0$

$2z - 6 = 0$

doplnit □ : $f = (x+1)^2 + (y+2)^2 + (z-3)^2 - (1+4+9)$

$= \|\underset{\sim}{x} - (-1, -2, 3)\|^2 - 14$

$\to +\infty$ pro $\|\underset{\sim}{x}\| \to +\infty$.

$\Rightarrow \exists$ glob. minimum

$\to (-1, -2, 3)$ je glob. min.

1.8. $f=(x^2+y^2)e^{-(x^2+y^2)}$ ; $M=\mathbb{R}^2$

... stačí vyšetřit $\varphi(t)=te^{-t}$ na $\langle 0,+\infty)$

$t=0$ .. glob. min $(x,y)=(0,0)$

$t=1$ glob. max $x^2+y^2=1$

1.9. $f=x^2+y^2+z^2$; $M=\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1\right\}$

$a>b>c>0$.

($\alpha$) uvnitř: $\nabla f=(2x,2y,2z)=0$

bod $(0,0,0)$ .. zjevně glob. MIN.

($\beta$) hranice: $\partial M=\{g=0\}$; $g=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$.

(i) $\nabla g=0$: nikde na $g=0$.

(ii) $\nabla f=\lambda\nabla g$ $\quad$ $2x=\lambda 2x\cdot\frac{1}{a^2}$

$2y=\lambda 2y\cdot\frac{1}{b^2}$

$2z=\lambda 2z\cdot\frac{1}{c^2}$.

1. rce: $x\neq 0 \Rightarrow \lambda=a^2$; 2.,3. rce $\Rightarrow$ $2y=2y\frac{a^2}{b^2}$

nutně $y=z=0$. $\quad 2z=2z\frac{a^2}{c^2}$,

$\leadsto$ podezřelé body $(\pm a,0,0)$ : MAX.

analogicky $(0,\pm b,0)$

$(0,0,\pm c)$.

**1.10.** ① selským rozumem:

$M$ ... elipsa s poloosami $1$, $\frac{1}{2}$

$f$ ... (vzdálenost od počátku$)^2$.

$\Rightarrow$ $(0, \pm\frac{1}{2})$ ... MIN

$(\pm 1, 0)$ ... MAX

② parametrizací $\begin{array}{l} x = \cos t \\ y = \frac{1}{2}\sin t \end{array}$ ; $t \in [0, 2\pi]$.

$$\varphi(t) = f\left(\cos t, \tfrac{1}{2}\sin t\right) = \cos^2 t + \tfrac{1}{4}\sin^2 t$$

$$= 1 - \tfrac{3}{4}\sin^2 t .$$

MAX: $\sin^2 t = 0$ ; $t = 0, \pi, (2\pi)$ : $(\pm 1, 0)$

MIN : $\sin^2 t = 1$ ; $t = \frac{\pi}{2}, \frac{3\pi}{2}$ : $\left(0, \pm\frac{1}{2}\right)$

③ Lagrange (i) $\nabla g = 0$ : nikde na $\{g = 0\}$

$$g = x^2 + 4y^2 - 1$$

(ii) $\nabla f = \lambda \nabla g$

$$2x = \lambda 2x$$

$$2y = \lambda 4y$$

1. rce:

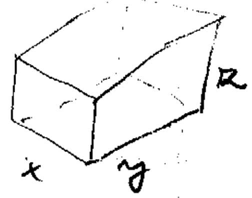
$x \neq 0 \Rightarrow \lambda = 1$ ; 2. rce $\Rightarrow y = 0$

analogicky $y \neq 0 \Rightarrow x = 0$ ;

tj: $\left(0, \pm\frac{1}{2}\right)$ ... jediné podezřelé body

$(\pm 1, 0)$

1.11. [Lagr.]




$$f = xy + 2(xz + yz) = xy + 2xz + 2yz$$

$$\Gamma = \{ xyz = 32 \}.$$

? $f$ shora neomezená na $\Gamma$:

$$z = \frac{32}{n} \; ; \quad f \geq xy = \frac{32}{z} = n$$

$n \in \mathbb{N}$ libovolné!

? podezřelé body: $\nabla g = (yz, xz, xy) \neq \underline{0}$ na $\Gamma$.

$x, y, z \neq 0$.

$$\nabla f = \lambda \nabla g$$

$$y + 2z = \lambda \cdot yz$$

$$x + 2z = \lambda \cdot xz$$

$$2x + 2y = \lambda \cdot xy$$

(1)−(2): $y - x = \lambda z (y - x)$

(a) $y \neq x$: $\lambda z = 1$: $y + 2z = y$

$z = 0$ spor

→ (b) $y = x$: (3): $4x = \lambda x^2$

$\underline{4 = \lambda x}$

(2): $x + 2z = 4z$

$x = 2z$

celkem: $(x, y, z) = (2t, 2t, t)$.

$\Gamma$: $4t^3 = 32$ ; $\underline{t = 2}$.

$(4,4,2)$ … podezřelý;

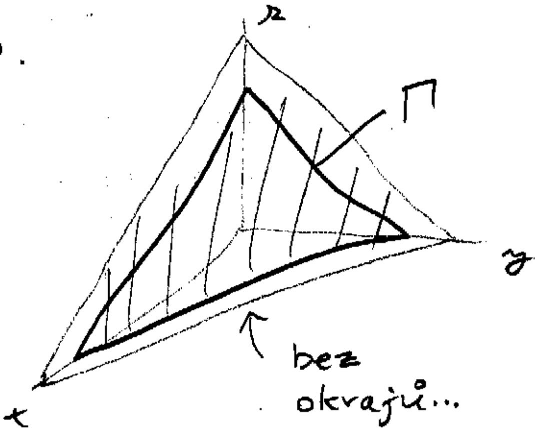
$f(4,4,2) = 16 + 2\cdot 8 + 2 \cdot 8 = 48$.

vol $n > 48 \Rightarrow f > 48$ pro $z \leq \frac{32}{n}$

**1.12.** $f = 50\, x^{2/5} y^{1/5} z^{1/5}$

$\Pi = \{ 80x + 12y + 10z = 24000 \} \cap \{ x, y, z > 0 \}$.

$g = 80x + 12y + 10z - 24000$.




$\nabla f = \lambda \nabla g$

$$20\, x^{-3/5} y^{1/5} z^{1/5} = \lambda \cdot 80$$
$$10\, x^{2/5} y^{-4/5} z^{1/5} = \lambda \cdot 12,$$
$$10\, x^{2/5} y^{1/5} z^{-4/5} = \lambda \cdot 10$$

$1:2 \Rightarrow \quad 2 \cdot x^{-1} y^{1} = 80/12 = \frac{20}{3},$

$$\underline{y = \frac{10}{3} x}$$

$1:3 \Rightarrow \quad 2 \cdot x^{-1} z = 8$

$$\underline{z = 4x}$$

$$80x + \underbrace{12 \cdot \frac{10}{3}}\, x + 10 \cdot 4x = 24000$$

$$160x = 24000$$

$$\boxed{\begin{aligned} x &= 150 \\ y &= 500 \\ z &= 600 \end{aligned}} \quad A$$

?? Extrém: $\tilde{M} = \Pi \cap \{x \geq \varepsilon\} \cap \{y \geq \varepsilon\} \cap \{z \geq \varepsilon\}$

$(x,y,z) \in \tilde{M} \setminus \Pi \Rightarrow$ buď $x$ nebo $y$ nebo $z$ $\leq \varepsilon$

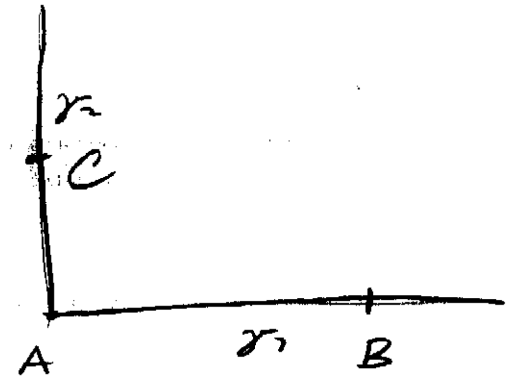
$\Rightarrow f$ malé. $(< f(A).)$

$\tilde{M}$ : kompakt: $\exists$ extrém; kraj nepadá v úvahu

A jediný podezřelý bod.

1.13. $f=(x+y)e^{-2x-3y}$

$M=\{x\geq 0\}\cap\{y\geq 0\}$.




($\alpha$) $y=0;\ x\geq 0$

$\varphi(t)=f(t,0);\quad t\in[0,\infty)$.

$=t\,e^{-2t};$

$\varphi'(t)=e^{-2t}(1-2t)$.

podezřelé body: $t=0$ & $t=\frac{1}{2}$

($\beta$) $x=0;\ y\geq 0$

$\varphi(t)=t\,e^{-3t}$

$\varphi'(t)=e^{-3t}(1-3t);\quad t=0$ & $t=\frac{1}{3}$

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($\gamma$): uvnitř: $\nabla f=\underset{\sim}{0}$.

$\partial_x f=e^{-2x-3y}\big(1-2(x+y)\big)$

$\partial_y f=e^{-2x-3y}\big(1-3(x+y)\big)$

---

$x+y=0$

$\rightsquigarrow$ žádné řešení na vnitřku!! $(x>0$ & $y>0)$.

$f(A)=0$ min – globální.

$f(B)=\frac{1}{2}e^{-1}$ max

$f(C)=\frac{1}{3}e^{-1}$

Globální?? $\tilde{M}=M\cap\{x+y\geq K\}$.

$f=(x+y)e^{-(x+y)}\cdot e^{-x-2y}$

$\leq(x+y)e^{-(x+y)}<\varepsilon$ pro

$x+y\geq K$.