① $$\frac{1}{3x^2-2x-1} = \frac{A}{x-1} + \frac{B}{3x+1} \quad \Big/ \cdot 3x^2-2x-1$$

$$1 = A(3x+1) + B(x-1)$$

$$\Rightarrow \quad A = \frac{1}{4}, \quad B = -\frac{3}{4}$$

$$I = \frac{1}{4}\int \frac{dx}{x-1} - \frac{3dx}{3x+1} = \frac{1}{4}\ln\left|\frac{x-1}{3x+1}\right| \qquad x \in (-\infty, -\tfrac{1}{3})$$

m.x.

$$(-\tfrac{1}{3}, 1)$$

$$(1, +\infty).$$

② $$\frac{1}{2}\int \frac{2x}{x^4-2x^2-1} = \frac{1}{2}\int \frac{dy}{y^2-2y-1};$$

$$y^2-2y-1 = (y-1)^2-2 = (y-(1+\sqrt{2}))(y-(1-\sqrt{2}))$$

$$\frac{1}{y^2-2y-1} = \frac{A}{y-(1+\sqrt{2})} + \frac{B}{y-(1-\sqrt{2})}; \quad A = -B = \frac{1}{2\sqrt{2}}$$

$$I = \frac{1}{4\sqrt{2}}\int \frac{1}{y-(1+\sqrt{2})} - \frac{1}{y-(1-\sqrt{2})}\,dy = \frac{1}{4\sqrt{2}}\ln\left|\frac{y-(1+\sqrt{2})}{y-(1-\sqrt{2})}\right|$$

$$= \frac{1}{4\sqrt{2}}\ln\left|\frac{x^2-(1+\sqrt{2})}{x^2-(1-\sqrt{2})}\right|; \qquad y \in (-1, 1-\sqrt{2})$$

m.x.

$$(1-\sqrt{2}, 1+\sqrt{2})$$

$$(1+\sqrt{2}, +\infty)$$

$$x \in (-\infty, -\sqrt{1+\sqrt{2}})$$

$$(-\sqrt{1+\sqrt{2}}, \sqrt{1+\sqrt{2}})$$

$$(\sqrt{1+\sqrt{2}}, +\infty).$$

③ $$\frac{x+1}{x^2+x+1} = \frac{1}{2}\,\frac{2x+1}{x^2+x+1} + \frac{1}{2}\cdot\frac{1}{x^2+x+1}\,;$$

$$\int \frac{2x+1}{x^2+x+1}\,dx \left|\begin{array}{l} y = x^2+x+1 \\ dy = (2x+1)\,dx \end{array}\right\} = \int \frac{dy}{y} = \ln|y|$$

$$= \ln(x^2+x+1)\,; \quad x \in \mathbb{R}.$$

$$x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4}\left[\left(\frac{2x+1}{\sqrt{3}}\right)^2+1\right];$$

$$\int \frac{dx}{x^2+x+1} = \frac{4}{3}\int \frac{dx}{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1} \left|\begin{array}{l} \frac{2x+1}{\sqrt{3}} = y \\ \frac{2}{\sqrt{3}}\,dx = dy \end{array}\right\} = \frac{2}{\sqrt{3}}\int \frac{dy}{y^2+1}$$

$$= \frac{2}{\sqrt{3}}\,\mathrm{arctg}\,y = \frac{2}{\sqrt{3}}\,\mathrm{arctg}\left(\frac{2x+1}{\sqrt{3}}\right); \quad x \in \mathbb{R}.$$

m.x.

celkem: $$I = \frac{1}{2}\ln(x^2+x+1) + \frac{1}{\sqrt{3}}\,\mathrm{arctg}\left(\frac{2x+1}{\sqrt{3}}\right); \quad x \in \mathbb{R}$$

④ $$\frac{x^5}{x^2+x-2} = x^3 - x^2 + 3x - 5 + \frac{11x-10}{(x-1)(x+2)}\,;$$

$$\frac{11x-10}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}\,; \quad A = \frac{1}{3}\,; \quad B = \frac{32}{3}$$

$$I = \int x^3 - x^2 + 3x - 5 + \frac{1}{3}\cdot\frac{1}{x-1} + \frac{32}{3}\cdot\frac{1}{x+2}\,dx$$

$$= \frac{x^4}{4} - \frac{x^3}{3} + \frac{3}{2}x^2 - 5x + \frac{1}{3}\ln|x-1| + \frac{32}{3}\ln|x+2|$$

m.x.

$x \in (-\infty, -2)$

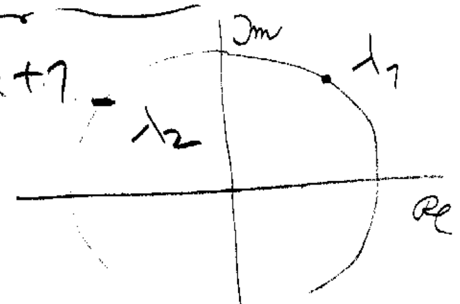
$(-2, 1)$

$(1, +\infty)$.

(5) $x^4+1 = \underbrace{(x-\lambda_1)(x-\overline{\lambda_1})}_{x^2-\sqrt{2}x+1} \cdot \underbrace{(x-\lambda_2)(x-\overline{\lambda_2})}_{x^2+\sqrt{2}x+1}$;

$\lambda_1 = \frac{1}{\sqrt{2}}(1+i)$

$\lambda_2 = \frac{1}{\sqrt{2}}(-1+i)$



jiný způsob: $x^4+1 = (x^2+1)^2 - 2x^2 = (x^2+1)^2 - (\sqrt{2}x)^2$

$[a^2-b^2=(a+b)(a-b)]$ $= (x^2+1+\sqrt{2}x)(x^2+1-\sqrt{2}x)$

$$\frac{1}{x^4+1} = \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1} ;$$

$A = -C = \frac{1}{2\sqrt{2}}$; $B = D = \frac{1}{2}$ m.x.

$$\frac{\frac{1}{2\sqrt{2}}x \pm \frac{1}{2}}{x^2 \pm \sqrt{2}x+1} = \frac{1}{4\sqrt{2}} \cdot \frac{2x \pm \sqrt{2}}{x^2 \pm \sqrt{2}x+1} + \frac{1}{4} \cdot \frac{1}{x^2 \pm \sqrt{2}x+1} ;$$

$$\int \frac{2x \pm \sqrt{2}}{x^2 \pm \sqrt{2}x+1} = \ln(x^2 \pm \sqrt{2}x+1) ; \quad x \in \mathbb{R}$$

$$\int \frac{1}{x^2 \pm \sqrt{2}x+1} = \int \frac{2}{(\sqrt{2}x \pm 1)^2+1} = \sqrt{2} \cdot \operatorname{arctg}(\sqrt{2}x \pm 1) ; \quad x \in \mathbb{R}$$

celkem: $I = \int \frac{dx}{x^4+1} = \int \frac{\frac{1}{2\sqrt{2}}x + \frac{1}{2}}{x^2+\sqrt{2}x+1} - \frac{\frac{1}{2\sqrt{2}}x - \frac{1}{2}}{x^2-\sqrt{2}x+1} \, dx$

$$= \frac{1}{4\sqrt{2}} \ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right) + \frac{1}{2\sqrt{2}} \operatorname{arctg}(\sqrt{2}x+1)$$

$$+ \frac{1}{2\sqrt{2}} \operatorname{arctg}(\sqrt{2}x-1) ; \quad x \in \mathbb{R}.$$

⑥ $\frac{1}{2}\int \frac{2x}{x^8-1}\,dx \quad \left| \begin{array}{l} x^2=y \\ 2x\,dx=dy \end{array} \right\} = \frac{1}{2}\int \frac{dy}{y^4-1}$

$$y^4-1=(y^2+1)(y^2-1)=(y^2+1)(y+1)(y-1)$$

$$\frac{1}{y^4-1}=\frac{Ay+B}{y^2+1}+\frac{C}{y+1}+\frac{D}{y-1}\ ;\quad A=0;\ B=-\frac{1}{2}$$

$$C=-\frac{1}{4},\ D=\frac{1}{4}$$

m.x.

$$=\frac{1}{2}\int -\frac{1}{2}\cdot\frac{1}{1+y^2}-\frac{1}{4}\cdot\frac{1}{y+1}+\frac{1}{4}\cdot\frac{1}{y-1}\,dy$$

$$=-\frac{1}{4}\operatorname{arctg} y+\frac{1}{8}\ln\left|\frac{y-1}{y+1}\right| \ ;\quad y\in(-\infty,-1)$$

$$(1,+\infty)$$

$$(-1,1)$$

$$=-\frac{1}{4}\operatorname{arctg} x^2+\frac{1}{8}\ln\left|\frac{x^2-1}{x^2+1}\right| \ ;\quad x\in(-\infty,-1)$$

m.x.

$$(-1,1)$$

$$(1,+\infty).$$

⑦ $\frac{1}{4}\int\frac{4x^3}{x^8+3}\,dx \quad \left| \begin{array}{l} x^4=y \\ 4x^3=dy \end{array} \right\} = \frac{1}{4}\int\frac{dy}{y^2+3}$

$$=\frac{1}{4\cdot 3}\int\frac{dy}{\left(\frac{y}{\sqrt{3}}\right)^2+1} \quad \left| \begin{array}{l} y=\sqrt{3}\,t \\ dy=\sqrt{3}\cdot dt \end{array} \right\} = \frac{1}{4\sqrt{3}}\int\frac{dt}{t^2+1}$$

$$=\frac{1}{4\sqrt{3}}\operatorname{arctg} t=\frac{1}{4\sqrt{3}}\operatorname{arctg}\left(\frac{x^4}{\sqrt{3}}\right) \ ;\quad x\in\mathbb{R}$$

m.x.

(8) $$\int \frac{x^3}{(x-1)^{100}}\,dx \;\left|\; \begin{matrix} x-1=y \\ dx=dy \end{matrix} \right\} = \int \frac{(y+1)^3}{y^{100}}\,dy$$

$$= \int \frac{y^3+3y^2+3y+1}{y^{100}}\,dy = \int y^{-100} + 3y^{-99} + 3y^{-98} + y^{-97}\,dy$$

$$= -\frac{1}{99y^{99}} - \frac{3}{98y^{98}} - \frac{3}{97y^{97}} - \frac{1}{96y^{96}}, \quad y \in (-\infty, 0) \; (0, +\infty)$$

$$= -\frac{1}{99(x-1)^{99}} - \frac{3}{98(x-1)^{98}} - \frac{3}{97(x-1)^{97}} - \frac{1}{96(x-1)^{96}}, \quad x \in (-\infty, 1) \; (1, +\infty).$$

(9) $$x^5+x^4-2x^3-2x^2+x+1 = (x-1)^2(x+1)^3$$

$$\frac{1}{(x-1)^2(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{x-1} + \frac{E}{(x-1)^2}$$

$$A = \frac{3}{16},\; B = \frac{1}{4},\; C = \frac{1}{4},\; D = -\frac{3}{16},\; E = \frac{1}{8} \quad \text{m.x.}$$

$$\int \frac{dx}{(x-1)^2(x+1)^3} = \frac{3}{16}\ln\left|\frac{x+1}{x-1}\right| - \frac{1}{4(x+1)} - \frac{1}{8(x+1)^2} - \frac{1}{8(x-1)}$$

$$x \in (-\infty, -1) \; (-1, +1) \; (1, +\infty)$$