10. $\int \frac{1}{\sqrt{(x-a)(b-x)}}$; notăm $a<b \Rightarrow x\in(a,b)$

$-\;a\;+\;b\;-$ $\qquad$ $-\;b\;+\;a\;-$

$$\frac{1}{\sqrt{(x-a)(b-x)}} = \frac{1}{(x-a)}\sqrt{\frac{x-a}{b-x}} \quad ; \quad t=\sqrt{\frac{x-a}{b-x}} \qquad t^2(b-x)=x-a$$

$$t^2=\frac{x-a}{b-x} \qquad t^2b-t^2x=x-a$$

$$x(1+t^2)=t^2b+a$$

$$x-a=\frac{t^2b+\cancel{a}-\cancel{a}-at^2}{1+t^2} = \frac{t^2(b-a)}{1+t^2} \qquad dx=\frac{-2t(a-b)}{(t^2+1)^2} \qquad x=\frac{t^2b+a}{1+t^2} = b+\frac{a-b}{t^2+1}$$

$$\rightarrow \cancel{\frac{1+t^2}{t^2(b-a)}}\cdot t\cdot\frac{+2t(a-b)}{(t^2+1)^2}dt = \frac{2}{t^2+1} = (2\,\text{arctg}\, t)' ;$$

$$\boxed{2\,\text{arctg}\sqrt{\frac{x-a}{b-x}}}\ \checkmark_{m.x.}$$

11. $\int\frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}dx$;

$$\frac{1-\sqrt{\frac{x-1}{x+1}}}{1+\sqrt{\frac{x-1}{x+1}}} \; ; \quad t=\sqrt{\frac{x-1}{x+1}} ; \quad t^2=\frac{x-1}{x+1}$$

$$t^2(x+1)=x-1$$

$$t^2x+t^2=x-1$$

$$x=\frac{1+t^2}{1-t^2}$$

$$dx=\frac{2t(1-t^2)+2t(1+t^2)}{(1-t^2)^2}=\frac{4t}{(1-t^2)^2}\checkmark_{m.x.}$$

$$\int\frac{1-t}{1+t}\cdot\frac{4t}{(1-t^2)^2}dt=\int\frac{4t}{(1+t)^3(1-t)}dt$$

$$\frac{4t}{(1+t)^3(1-t)}=\frac{A}{1-t}+\frac{B}{(1+t)^3}+\frac{C}{(1+t)^2}+\frac{D}{1+t}$$

$$4t = A(1+t)^3+B(1-t)+C(1-t^2)+D(1+t)^2(1-t^2)$$

$t=1$: $4=A\cdot 8$; $A=\frac{1}{2}$

$t=-1$: $-4=2B$; $B=-2$

$$4 = 3A(1+t)^2-B-2Ct+D\left(1-t^2-2t(1+t)\right)$$

$$+D(1-2t-3t^2)$$

$t=-1$ $\quad 4 = 2+2C$; $\quad C=1$

$2C=2$

$$0 = 6A(1+t) - 2C + D(-2-6t)$$

$x \in (1,+\infty)$.

$t=-1$: $2C = D(4)$; $D = \frac{1}{2}$

$$\frac{4t}{(1+t)^3(1-t)} = \frac{\frac{1}{2}}{1-t} - \frac{2}{(1+t)^3} + \frac{1}{(1+t)^2} + \frac{\frac{1}{2}}{1+t} \quad \text{m.x.}$$

$$= \left(\frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| + \frac{1}{(1+t)^2} - \frac{1}{1+t}\right)' \checkmark \ \text{m.x.}$$

$t \in (-\infty,-1)$
$(-1,1)$
$(1,+\infty)$

12. $\int \sqrt[4]{\frac{x-4}{x+4}}\,dx$; $t = \sqrt[4]{\frac{x-4}{x+4}}$, $t^4 = \frac{x-4}{x+4}$

$t^4x + 4t^4 = x - 4$

$$x = \frac{4(1+t^4)}{1-t^4}$$

$$dx = \frac{32t^3}{(1-t^4)^2} \quad \text{m.x.}$$

$$\int \frac{32t^4}{(1-t^4)^2} =: I_t^{(1)}$$

$$\frac{32t^4 \pm 32}{(1-t^4)^2} = \frac{32}{t^4-1} + \frac{32}{(1-t^4)^2}$$

$// \ I_t$

per-partes:

$$\int \frac{dt}{1-t^4} = \frac{t}{1-t^4} - \int \frac{4t^4}{(1-t^4)^2} \ ; \quad \boxed{\frac{8t}{1-t^4} - 8\int \frac{dt}{1-t^4}} \quad \text{viz č. 6}$$

$u' = 1 \quad u = t$

$v = \frac{1}{1-t^4} \quad v' = \frac{4t^3}{(1-t^4)^2}$

$$\boxed{I_t = \frac{8t}{1-t^4} - 4\,\text{arctg}\,t + 2\ln\left|\frac{t-1}{t+1}\right|} \ \checkmark \quad \text{m.x}$$

13. $\int \frac{1-\sqrt{x+1}}{1-\sqrt[3]{x+1}}$; $t = \sqrt[6]{x+1}$, $t^6 = x+1$; $x = t^6 - 1$, $dx = 6t^5\,dt$

$$\int \frac{1-t^3}{1-t^2}\cdot 6t^5\,dt = \int \frac{6t^5(t^2+t+1)}{t+1}\,dt = -6\ln(t+1) + \frac{6}{7}t^7 + \frac{6}{5}t^5 - \frac{3}{2}t^4 + 2t^3 - 3t^2 + 6t$$

m.x.

$$\frac{t^5(t^2+t+1)}{t+1} = t^6 + t^4 - t^3 + t^2 - t + 1 - \frac{1}{t+1}$$

(14) $$\int \frac{x}{\sqrt[4]{x^3(1-x)}}\,dx = \int \sqrt[4]{\frac{x}{1-x}}\,dx \quad \Bigg| \quad t = \sqrt[4]{\frac{x}{1-x}}, \quad x = \frac{t^4}{1+t^4}\,;\; dx = \frac{4t^3}{(1+t^4)^2}$$

máme $x \in (0,1)$

m.x.

$$= \int \frac{4t^4}{(t^4+1)^2}\,dt \,; \qquad \underline{\underline{\text{viz}}}: \int \frac{dt}{t^4+1} = \frac{t}{t^4+1} + \int \frac{4t^4}{t^4+1}\,dt$$

viz č. (5)

m.x.

(15) $$\int \frac{x\,dx}{(1+x)(1-x^3)\sqrt{\frac{1-x}{1+x}}} \quad \Bigg| \quad \frac{1}{t} = \sqrt{\frac{1-x}{1+x}}\,; \quad x = \frac{1-t^2}{1+t^2}$$

$$dx = \frac{-4t\,dt}{(t^2+1)^2}\,; \quad 1-x^3 = \frac{2t^2(t^4+3)}{(t^2+1)^3}$$

$$1+x = \frac{2}{t^2+1}\,.$$

m.x.

$$= \int \frac{1-t^2}{\cancel{1+t^2}} \cdot \frac{t^2+1}{\cancel{2}} \cdot \frac{\cancel{(t^2+1)^3}}{\cancel{2}t^2(t^4+3)} \cdot \frac{1}{\cancel{t}} \cdot \frac{(-4\cancel{t})}{\cancel{(t^2+1)^2}}\,dt$$

$$= \int \frac{t^4-1}{t^2(t^4+3)}\,dt \,;$$

rozklad: $t^2 = y$: $$\frac{y^2-1}{y(y^2+3)} = -\frac{1}{3y} + \frac{4y}{3(y^2+3)} \,;$$

$$\frac{t^4-1}{t^2(t^4+3)} = \underbrace{-\frac{1}{3t^2}}_{\text{máme}} + \frac{4}{3} \cdot \underbrace{\frac{t^2}{t^4+3}}_{I_1}\,.$$

$$I_1 = \int \frac{t^2}{t^4+3}\,dt = \frac{1}{3} \int \frac{t^2\,dt}{\left(\frac{t}{\sqrt[4]{3}}\right)^4+1} \quad \Bigg| \quad t = x\cdot\sqrt[4]{3}, \quad t^2 = x^2\sqrt{3}, \quad dt = \sqrt[4]{3}\,dx$$

$$= \frac{1}{3}\cdot\sqrt{3}\cdot\sqrt[4]{3} \int \frac{x^2}{x^4+1}\,dx = \frac{1}{\sqrt[4]{3}} \cdot J\,.$$

pokračování (15): teď znovu $J = \int \frac{x^2\,dx}{x^4+1}$;

víme (viz (5)): $x^4+1 = (x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$

$$\frac{x^2}{x^4+1} = \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1} \quad m.$$

$$x^2 = (Ax+B)(x^2-\sqrt{2}x+1) + (Cx+D)(x^2+\sqrt{2}x+1)$$

$$-A = C = \frac{1}{2\sqrt{2}}\,; \quad B = D = 0. \quad m.x.$$

$$\int \frac{x\,dx}{x^2-\sqrt{2}x+1} = \frac{1}{2}\underbrace{\int \frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}\,dx}_{\ln(x^2-\sqrt{2}x+1)} + \frac{1}{\sqrt{2}}\underbrace{\int \frac{dx}{x^2-\sqrt{2}x+1}}_{K}\,;$$

$$K = \int \frac{dx}{\left(x-\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}} = \int \frac{2\,dx}{(\sqrt{2}x-1)^2+1} = \frac{2}{\sqrt{2}}\operatorname{arctg}(\sqrt{2}x-1)\,; \; x\in\mathbb{R}.$$

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tedy: $$\int \frac{x\,dx}{x^2-\sqrt{2}x+1} = \frac{1}{2}\ln(x^2-\sqrt{2}x+1) + \operatorname{arctg}(\sqrt{2}x-1)\,; \quad x\in\mathbb{R}$$

a celkem:

$$J = \frac{1}{4\sqrt{2}}\ln\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1} + \frac{1}{2\sqrt{2}}\operatorname{arctg}(\sqrt{2}x+1)$$

$$+\frac{1}{2\sqrt{2}}\operatorname{arctg}(\sqrt{2}x-1)\,; \quad x\in\mathbb{R}$$