(16) $\displaystyle\int \frac{dx}{x\sqrt{x^2+x+1}}$ | $\sqrt{x^2+x+1} = t - x$ $\quad x \in \mathbb{R} \Leftrightarrow t \in (-\frac{1}{2}, +\infty)$

$x^2+x+1 = t^2 - 2tx + x^2 \quad x=0:\ t=1.$

$x = \dfrac{t^2-1}{2t+1}\ ;\ dx = \dfrac{2(t^2+t+1)}{(2t+1)^2}\,dt$ m.x.

$$\sqrt{x^2+x+1} = t - x = t - \frac{t^2-1}{2t+1} = \frac{t^2+t+1}{2t+1}.$$

$$= \int \frac{2t+1}{t^2-1} \cdot \frac{2t+1}{t^2+t+1} \cdot \frac{2(t^2+t+1)}{(2t+1)^2}\,dt = \int \frac{2\,dt}{t^2-1} = \ln\left|\frac{t-1}{t+1}\right|$$

$t \in (-\infty, -1)$, $(-1, +1)$, $(1, +\infty)$.

celkem: $\displaystyle = \ln\left|\frac{\sqrt{x^2+x+1}+x-1}{\sqrt{x^2+x+1}+x+1}\right|\ ;\ x \in (-\infty, 0),\ (0, +\infty).$

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(17) $\displaystyle\int \frac{x}{\sqrt{x^2+x+1}}\,dx$ | $\sqrt{x^2+x+1} = t - x\ ;$ viz již (16) výp.:

$$= \int \frac{t^2-1}{2t+1} \cdot \frac{2t+1}{t^2+t+1} \cdot \frac{2(t^2+t+1)}{(2t+1)^2}\,dt = \int \frac{2(t^2-1)}{(2t+1)^2}\,dt$$

$$\frac{2(t^2-1)}{(2t+1)^2} = \frac{1}{2} - \frac{1}{2t+1} - \frac{3}{2(2t+1)^2}\ ;\ \text{m.x.}$$

$$= \frac{t}{2} - \frac{1}{2}\ln|2t+1| + \frac{3}{4(2t+1)}\ ;\quad t \in (-\infty, -\tfrac{1}{2}),\ (\tfrac{1}{2}, +\infty).$$

$$= \frac{1}{2}\left(\sqrt{x^2+x+1}+x\right) - \frac{1}{2}\ln\left(2\left(\sqrt{x^2+x+1}+x\right)+1\right)$$

$$- \frac{3}{2} \cdot \frac{1}{\left(2\left(\sqrt{x^2+x+1}+x\right)+1\right)^2}\ ;\quad x \in \mathbb{R}$$

(18) $\displaystyle\int \frac{x^2+1}{x\sqrt{x^4+1}}\,dx = \frac{1}{2}\int \frac{x^2+1}{x^2\sqrt{(x^2)^2+1}}\cdot 2x\,dx$ $\Big|$ $x^2 = y$, $2x\,dx = dy$

$\displaystyle = \frac{1}{2}\int \frac{y+1}{y\sqrt{y^2+1}}\,dy$ $\Big|$ $\sqrt{y^2+1} = t - y$; $y \in \mathbb{R} \Leftrightarrow t \in (0,+\infty)$

$y^2+1 = t^2 - 2ty + y^2$ $\quad$ $y=0 : t=1$

$\displaystyle y = \frac{t^2-1}{2t}$; $\displaystyle dy = \frac{t^2+1}{2t^2}\,dt$

$\displaystyle y+1 = \frac{t^2+2t-1}{2t}$; $\displaystyle \sqrt{y^2+1} = \frac{t^2+1}{2t}$

$$= \frac{1}{\cancel{2}}\int \frac{t^2+2t-1}{2t}\cdot\frac{\cancel{2t}}{t^2-1}\cdot\frac{\cancel{2t}}{\cancel{t^2+1}}\cdot\frac{\cancel{t^2+1}}{\cancel{2t^2}}\,dt = \int \frac{t^2+2t-1}{2t(t^2-1)}\,dt$$

$$\frac{t^2+2t-1}{2t(t^2-1)} = \frac{1}{2t} + \frac{1}{2(t-1)} - \frac{1}{2(t+1)} \; ; \quad = \frac{1}{2}\ln\left|\frac{t(t-1)}{t+1}\right|$$

$t \in (-\infty,-1)$, $(-1,0)$, $(0,1)$, $(1,+\infty)$.

⇝ výsledek do ~~sp~~ své nezmění: $x \in (-\infty,0)$, $(0,+\infty)$.

18 – jinak $\displaystyle \frac{1}{2}\int \frac{y+1}{y\sqrt{y^2+1}}$ $\Big|$ $y = \sinh t$, $dy = \cosh t\,dt$ $\quad$ $\sqrt{y^2+1} = \sqrt{\sinh^2 t+1} = \cosh t$

$$= \frac{1}{2}\int \frac{\sinh t+1}{\sinh t\cdot\cancel{\cosh t}}\cdot\cancel{\cosh t}\,dt = \frac{1}{2}\int\left(1+\frac{1}{\sinh t}\right)dt$$

$\displaystyle = \int \frac{1}{2}\left(1+\frac{2}{e^t-e^{-t}}\right)dt$ $\Big|$ $e^t = u$, $t = \ln u$, $\displaystyle dt = \frac{1}{u}\,du$

$$= \int \frac{1}{2}\left(1+\frac{2}{u-\frac{1}{u}}\right)\cdot\frac{du}{u} = \int \frac{u^2+2u-1}{2u(u^2-1)}\,du = \cdots$$ stejný integrál jako výše...

což není náhoda, neboť $\sinh^{-1}(x) = \ln\left(\sqrt{x^2+1}+x\right)$.

21 $\displaystyle\int \frac{dx}{(x^2+1)\sqrt{x^2-1}}$ $\left|\begin{array}{l} \sqrt{x^2-1} = t - x \\ x^2-1 = t^2-2tx+x^2 \\ x = \dfrac{t^2+1}{2t} \end{array}\right.$ $dx = \dfrac{t^2-1}{2t^2}\,dt$

$$\sqrt{x^2-1} = \frac{t^2-1}{2t}$$

$$x^2+1 = \frac{t^4+6t^2+1}{4t^2}.$$

$$= \int \frac{4t^2}{t^4+6t^2+1} \cdot \frac{2t}{\cancel{t^2-1}} \cdot \frac{\cancel{t^2-1}}{2\cancel{t^2}}\,dt$$

$$= \int \frac{4t\,dt}{t^4+6t^2+1} \left|\begin{array}{l} t^2 = u \\ 2t\,dt = du \end{array}\right\} = \int \frac{2\,du}{u^2+6u+1} = \frac{1}{2\sqrt{2}} \ln\left|\frac{u+3-2\sqrt{2}}{u+3+2\sqrt{2}}\right|$$

$$\frac{2}{u^2+6u+1} = \frac{-\frac{1}{2\sqrt{2}}}{u+3+2\sqrt{2}} + \frac{\frac{1}{2\sqrt{2}}}{u+3-2\sqrt{2}} \quad \text{m-x.}$$

21 - jiný postup: $x = \cosh t$; $t \in (0,+\infty) \Leftrightarrow x \in (1,+\infty)$

$dx = \sinh t\,dt$;

$x^2-1 = \cosh^2 t - 1 = \sinh^2 t$;

tj. $\sqrt{x^2-1} = \sinh t$; $t > 0$.

$$= \int \frac{\sinh t\,dt}{(\cosh^2 t+1)\sinh t} = \int \frac{dt}{\frac{1}{4}(e^t+e^{-t})^2+1} \left|\begin{array}{l} e^t = u \\ t = \ln u \\ dt = \dfrac{du}{u} \end{array}\right.$$

$$= \int \frac{4u}{u^4+6u^2+1} = \cdots$$ stejně jako výše;

což se dá čekat, neboť

$(\sinh)_{-1}(r) = \ln\left(r+\sqrt{r^2-1}\right)$

$r \geq 1$.

(22) $\displaystyle\int \frac{\sqrt{x^2+x+1}}{x^2+2x+1}\,dx$

$\sqrt{x^2+x+1} = t - x$

$$x = \frac{t^2-1}{2t+1}\,; \quad dx = \frac{2(t^2+t+1)}{(2t+1)^2}\,dt\,;$$

$$\sqrt{x^2+x+1} = \frac{t^2+t+1}{2t+1}$$

$$x^2+2x+1 = \left(\sqrt{x^2+x+1}\right)^2 + x = (t-x)^2 + x = \frac{t^2(t+2)^2}{(2t+1)^2}\,; \quad \text{m.x.}$$

$$= \int \frac{t^2+t+1}{\cancel{2t+1}} \cdot \frac{\cancel{(2t+1)^2}}{t^2(t+2)^2} \cdot \frac{2(t^2+t+1)}{(2t+1)^{\cancel{2}}}\,dt = \int \frac{2(t^2+t+1)^2}{t^2(t+2)^2(2t+1)}\,dt$$

$$= \int \frac{2}{2t+1} + \frac{1}{2(t+2)} - \frac{3}{2(t+2)^2} - \frac{1}{2t} + \frac{1}{2t^2}\,dt$$

$$= \ln|2t+1| - \frac{1}{2}\ln|t+2| + \frac{3}{2}\cdot\frac{1}{t+2} - \frac{1}{2}\ln|t| - \frac{1}{2t}\,;$$

... po substituci: $t = x + \sqrt{x^2+x+1}$ platí na $x \in (-\infty,-1)$ $(-1,+\infty)$.

(23) $\displaystyle\int \frac{dx}{(x+1)^5\sqrt{x^2+2x}}$

$\sqrt{x^2+2x} = t - x$

$\underline{x^2}+2x = t^2-2tx+\underline{x^2}$

$$dx = \frac{t(t+2)}{2(t+1)^2}$$

$$x = \frac{t^2}{2(t+1)}\,; \quad x+1 = \frac{t^2+2t+2}{2(t+1)}\,; \quad \sqrt{x^2+2x} = \frac{t(t+2)}{2(t+1)}$$

$$= \int \frac{32(t+1)^{\cancel{5}\,4}}{(t^2+2t+2)^5} \cdot \frac{\cancel{2(t+1)}}{\cancel{t(t+2)}} \cdot \frac{\cancel{t(t+2)}}{\cancel{2(t+1)^2}}\,dt = \int \frac{32(t+1)^4}{((t+1)^2+1)^5} \quad \Bigg|\ \begin{aligned} t+1 &= u \\ dt &= du \end{aligned}$$

$$= \int \frac{32u^4}{(u^2+1)^5} = 32\int \frac{1}{(u^2+1)^3} - \frac{2}{(u^2+1)^4} + \frac{1}{(u^2+1)^5}\,du\,;$$

... při rozkladu značme $u^2 = y$

... k integraci použijeme rekurentní vzorec pro $I_m = \displaystyle\int \frac{du}{(u^2+1)^m}$; viz př. (48)

L23 - jinak: $\sqrt{x(x+2)} = \sqrt{\frac{x+2}{x}x^2} = |x|\sqrt{\frac{x+2}{x}} = \pm x\sqrt{\frac{x+2}{x}}$

$x \in (0, +\infty)$

nebo $(-\infty, -2)$.

$$\sqrt{\frac{x+2}{x}} = t \qquad x = \frac{2}{t^2-1} \ \ldots$$

$$\frac{x+2}{x} = t^2 \qquad dx = \frac{-4t}{(t^2-1)^2}\,dt \qquad x+1 = \frac{t^2+1}{t^2-1}$$

$$\int \frac{dx}{(x+1)^5\sqrt{x(x+2)}} = \int \frac{dx}{(x+1)^5 x\sqrt{\frac{x+2}{x}}} = \int \frac{(t^2-1)^{\cancel{5}\,4}}{(t^2+1)^5}\cdot\frac{\cancel{t^2-1}}{2}\cdot\frac{(-4t)}{\cancel{(t^2-1)^2}}\,dt$$

$$= \int \frac{-2t(t^2-1)^4}{(t^2+1)^5}\,dt \ \left|\ \begin{matrix} t^2 = u \\ 2t\,dt = du \end{matrix}\right\} = \int \frac{-(u-1)^4}{(u+1)^5}\,du$$

$$= \int \frac{-1}{u+1} + \frac{8}{(u+1)^2} - \frac{24}{(u+1)^3} + \frac{32}{(u+1)^4} - \frac{16}{(u+1)^5}\,du$$

$$= -\ln|u+1| - \frac{8}{u+1} + \frac{12}{(u+1)^2} - \frac{32}{3(u+1)^3} + \frac{4}{(u+1)^4} \ \ldots$$