(36) $\int \frac{dx}{2+\sin x}$ $\Bigg|$ $t = \operatorname{tg}\frac{x}{2}$; $x \in (-\pi,\pi) \longleftrightarrow t \in \mathbb{R}$

$\sin x = \frac{2t}{1+t^2}$ $\quad dx = \frac{2}{1+t^2}\,dt$

$$= \int \frac{1}{2+\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt = \int \frac{dt}{t^2+t+1} = \frac{2}{\sqrt{3}} \operatorname{arctg}\left(\frac{2t+1}{\sqrt{3}}\right)$$

$t \in \mathbb{R}$.

oznaczmy $F_0(x) = \frac{2}{\sqrt{3}} \operatorname{arctg}\left(\frac{2\operatorname{tg}\frac{x}{2}+1}{\sqrt{3}}\right)$; x

stąd: $\int \frac{dx}{2+\sin x} = F_0(x)$; $x \in (-\pi,\pi)$; ogólniej

dla $x \in ((2k-1)\pi, (2k+1)\pi)$

$\forall k \in \mathbb{Z}$.

zauważmy: $\lim\limits_{x\to\pi-} F_0(x) = \frac{\pi}{\sqrt{3}}$; $\lim\limits_{x\to-\pi+} F_0(x) = -\frac{\pi}{\sqrt{3}}$.

połóż:
$$F_1(x) = \begin{cases} F_0(x); & x \in (-\pi,\pi) \\ \frac{\pi}{\sqrt{3}}; & x = \pi \\ F_0(x) + \frac{2\pi}{\sqrt{3}}; & x \in (\pi, 3\pi). \end{cases}$$

jest $F_1(x)$ ciągła w $(-\pi, 3\pi)$;

więc $\int \frac{dx}{2+\sin x} = F_1(x)$ w $(-\pi, 3\pi)$.

(37) $\displaystyle\int \frac{dx}{(2+\cos x)\sin x} = \int \frac{\sin x\, dx}{(2+\cos x)\sin^2 x}$ $\quad\Bigg|\quad \begin{aligned} &y = \cos x \\ &dy = -\sin x\, dx \\ &\sin^2 x = 1-\cos^2 x = 1-y^2. \end{aligned}$

$$= \int \frac{dy}{(y+2)(y^2-1)} = \int \frac{\frac{1}{3}}{y+2} - \frac{\frac{1}{2}}{y+1} + \frac{1/6}{y-1}$$

$$= \frac{1}{3}\ln|y+2| - \frac{1}{2}\ln|y+1| + \frac{1}{6}\ln|y-1| \quad ; \quad y \in (-\infty,-2),\ (-2,-1),\ (-1,1),\ (1,+\infty)$$

$$= \frac{1}{3}\ln(\cos x+2) - \frac{1}{2}\ln(\cos x+1) + \frac{1}{6}\ln(1-\cos x)$$

$$x \in (k\pi, (k+1)\pi).$$

„standardní postup": $t = \operatorname{tg}\frac{x}{2}$; … vede na $\displaystyle\int \frac{t^2+1}{t(t^2+3)}\,dt$

(38) $\displaystyle\int \frac{\sin x \cos x}{\sin x + \cos x}\,dx$ $\quad\Bigg|\quad \begin{aligned} &t = \operatorname{tg}\frac{x}{2}; \quad \sin x = \frac{2t}{1+t^2} \\ &dx = \frac{2}{1+t^2}\,dt \quad \cos x = \frac{1-t^2}{1+t^2} \end{aligned}$

$$= \int \frac{\frac{2t}{1+t^2}\cdot\frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt = \int \frac{4t(1-t^2)}{(t^2+1)^2(2t+1-t^2)}\,dt = \dots$$

$$\frac{4t(t^2-1)}{(t^2+1)^2(t^2-2t-1)} = \frac{A}{t-\sqrt{2}-1} + \frac{B}{t+\sqrt{2}-1} + \frac{Ct+D}{t^2+1} + \frac{Et+F}{(t^2+1)^2}$$

$$= \frac{1}{2\sqrt{2}}\left(\frac{1}{t-\sqrt{2}-1} - \frac{1}{t+\sqrt{2}-1}\right) - \frac{1}{t^2+1} + \frac{2t+2}{(t^2+1)^2}$$

$$\dots = \frac{1}{2\sqrt{2}}\ln\left|\frac{t-\sqrt{2}-1}{t+\sqrt{2}-1}\right| - \operatorname{arctg} t + \underbrace{\int \frac{2t+2}{(t^2+1)^2}\,dt}_{I}.$$

38 dokončení: $I = \int \frac{2t}{(t^2+1)^2}dt + 2\underbrace{\int \frac{dt}{(t^2+1)^2}}_{I_2} \doteq I_1 + 2I_2$

$I_1 = \left| \begin{array}{l} t^2 = u \\ 2t\,dt = du \end{array} \right\} = \int \frac{du}{(u+1)^2} = -\frac{1}{u+1} = -\frac{1}{t^2+1}$;

$I_2 = \left| \begin{array}{l} \text{viz indukční} \\ \text{vzorec; př. 48} \end{array} \right) = \frac{t}{2(t^2+1)} + \frac{1}{2}\cdot \operatorname{arctg} t$.

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celkem: $I = \frac{-1}{t^2+1} + \frac{t}{t^2+1} + \operatorname{arctg} t$;

a tedy: $\int \frac{4t(t^2-1)\,dt}{(t^2+1)^2(t^2-2t-1)} = \frac{1}{2\sqrt2}\ln\left|\frac{t-\sqrt2-1}{t+\sqrt2-1}\right| + \frac{t-1}{t^2+1} = G(t)$

$t \in (-\infty, 1-\sqrt2), (1-\sqrt2, 1+\sqrt2)$
$(1+\sqrt2, +\infty)$.

$F(x) = G\left(\operatorname{tg}\frac{x}{2}\right)$ je p.f. k $\frac{\sin x\cdot\cos x}{\sin x+\cos x}$ kde?

vyloučit: $\sin x = -\cos x \iff \operatorname{tg}\frac{x}{2} = 1 \pm \sqrt2$

$x = -\frac{\pi}{4} + k\pi$. ;

tj.: $\int \frac{\sin x\cos x}{\sin x+\cos x}dx = F(x)$ v $\left(-\pi, -\frac{\pi}{4}\right)$
$\left(-\frac{\pi}{4}, \frac{3\pi}{4}\right)$
$\left(\frac{3\pi}{4}, \pi\right)$.

protože $\lim\limits_{\substack{x\to\pi- \\ -\pi+}} F(x) = 0$, lze $F(x)$ dodefinovat zde 0

mozné a mít p.f. na

intervalech: $\left(-\frac{\pi}{4}+k\pi, \frac{3\pi}{4}+k\pi\right)$; $k \in \mathbb{Z}$.

(39) $\displaystyle\int \frac{\cos x \, dx}{2\sin x - 3\cos x + 6}$. $\Big|$ $t = \operatorname{tg}\frac{x}{2}$ $\quad \sin x = \frac{2t}{1+t^2}$

$\quad dx = \frac{2\,dt}{1+t^2} \quad \cos x = \frac{1-t^2}{1+t^2}$

$$= \int \frac{\frac{1-t^2}{1+t^2}}{\frac{4t}{1+t^2} - \frac{3(1-t^2)}{1+t^2} + 6} \cdot \frac{2\,dt}{1+t^2} = \int \frac{2(1-t^2)\,dt}{(t^2+1)(9t^2+4t+3)} \cdot$$

$$= \frac{1}{13}\int \frac{36t+44}{9t^2+4t+3}\,dt - \frac{1}{13}\int \frac{4t+6}{t^2+1}\,dt = \frac{1}{13}\left(I_1 - I_2\right).$$

$$I_2 = 2\ln(t^2+1) + 6\operatorname{arctg} t \,, \quad t \in (-\infty, +\infty).$$

$$I_1 = \underbrace{2\int \frac{18t+4}{9t^2+4t+3}\,dt}_{\ln(9t^2+4t+3)} + \underbrace{36\int \frac{dt}{9t^2+4t+3}}_{I_3}$$

---

ad I_3: $\quad 9t^2+4t+3 = \left(3t+\frac{2}{3}\right)^2 + \frac{23}{9} = \frac{23}{9}\left[\left(\frac{9t+2}{\sqrt{23}}\right)^2 + 1\right]$

$$I_3 = \frac{9}{23}\cdot\int \frac{dt}{\left(\frac{9t+2}{\sqrt{23}}\right)^2+1} = \underbrace{\frac{9}{23}\cdot\frac{\sqrt{23}}{9}}_{\frac{1}{\sqrt{23}}} \operatorname{arctg}\left(\frac{9t+2}{\sqrt{23}}\right)$$

---

m.x.

celkem: $\displaystyle = \frac{1}{13}\left\{2\ln\frac{(9t^2+4t+3)}{t^2+1} + \frac{1}{\sqrt{23}}\operatorname{arctg}\left(\frac{9t+2}{\sqrt{23}}\right) - 6\operatorname{arctg} t\right\}$

$t \in \mathbb{R}$.

→ dosadit $t = \operatorname{tg}\frac{x}{2}$. → $F(x)$; $x \in (-\pi, \pi)$

$$\lim_{\substack{x\to\pi- \\ -\pi+}} F(x) = \frac{1}{13}\left\{2\ln 9 + \frac{\pi}{2}\left(\frac{1}{\sqrt{23}} - 6\right)\right\}.$$

$-$

lze napojit!!

(40) $\int \frac{dx}{\sin x}$; $\Big| \; t = \operatorname{tg} \frac{x}{2} \; \ldots \Big\} = \int \frac{1}{\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\, dt$

$= \int \frac{dt}{t} = \ln|t| = \ln|\operatorname{tg} \frac{x}{2}|$; $\quad t \in (-\infty, 0), (0, +\infty)$

$\cdots \; x \in (k\pi, (k+1)\pi)$.

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(41) $\int \frac{\sin^2 x}{1+\sin^2 x}\, dx$ $\Big| \; t = \operatorname{tg} x, \; dx = \frac{dt}{1+t^2}$; $\quad \sin x = \frac{t}{\sqrt{1+t^2}}$, $\cos x = \frac{1}{\sqrt{1+t^2}}$

$= \int \frac{\frac{t^2}{1+t^2}}{1+\frac{t^2}{1+t^2}} \cdot \frac{dt}{1+t^2} = \int \frac{t^2\, dt}{(2t^2+1)(t^2+1)}$. m.x.

$= \int \frac{1}{t^2+1} - \frac{1}{2t^2+1}\, dt = \operatorname{arctg} t - \frac{1}{\sqrt{2}} \operatorname{arctg}(\sqrt{2}\, t)$;

$t \in (-\infty, \infty) \; \ldots \; x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

množení: $F_0(x) := \operatorname{arctg}(\operatorname{tg} \frac{x}{2}) - \frac{1}{\sqrt{2}} \operatorname{arctg}(\sqrt{2} \operatorname{tg} \frac{x}{2})$;

$f(x) = \frac{\sin^2 x}{1+\sin^2 x}$; zřejmě: $F_0'(x) = f(x)$; $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

---

definuji: $F_1(x) := \begin{cases} F_0(x); & x \in (-\frac{\pi}{2}, \frac{\pi}{2}) \\ \frac{\pi}{2}(1 - \frac{1}{\sqrt{2}}); & x = \frac{\pi}{2} \\ F_0(x) + \pi(1 - \frac{1}{\sqrt{2}}); & x \in (\frac{\pi}{2}, \frac{3\pi}{2}) \end{cases}$

$\Rightarrow \; F_1'(x) = f(x)$; $x \in (-\frac{\pi}{2}, \frac{3\pi}{2})$.

POZOR: $\operatorname{arctg}(\operatorname{tg} \frac{x}{2}) \neq x$ mimo $(-\frac{\pi}{2}, \frac{\pi}{2})$.

(42) $$\int \mathrm{tg}^5 x\,dx \;\left|\; \begin{matrix} t = \mathrm{tg}\,x \\ dx = \dfrac{dt}{1+t^2} \end{matrix} \right\} = \int \frac{t^5}{1+t^2}\,dt \qquad \sin^2 x + \cos^2 x = 1$$

$$= \int t^3 - t + \frac{t}{t^2+1}\,dt = \frac{t^4}{4} - \frac{t^2}{2} + \ln\sqrt{t^2+1} \;; \qquad \mathrm{tg}^2 x + 1 = \frac{1}{\cos^2 x}$$

$$\cos x \neq 0.$$

$$= \frac{1}{4}\mathrm{tg}^4 x - \frac{1}{2}\mathrm{tg}^2 x - \ln|\cos x| \quad \text{m.x.} \;; \quad x \in \left(\left(k-\tfrac{1}{2}\right)\pi, \left(k+\tfrac{1}{2}\right)\pi\right).$$

(43) $$\int \frac{1-\mathrm{tg}\,x}{1+\mathrm{tg}\,x}\,dx = \left|\; \begin{matrix} t = \mathrm{tg}\,x \\ dx = \dfrac{dt}{1+t^2} \end{matrix} \right\} = \int \frac{1-t}{(1+t)(1+t^2)}\,dt$$

$$= \int \frac{1}{t+1} - \frac{t}{t^2+1}\,dt = \ln\left|\frac{t+1}{\sqrt{t^2+1}}\right| \;; \quad t \in (-\infty, -1),\ (-1, +\infty). \quad \text{m.x.}$$

$$f(t) = \frac{1-\mathrm{tg}\,x}{1+\mathrm{tg}\,x} \;; \quad F_0(t) = \ln\left|\frac{\mathrm{tg}\,x + 1}{\sqrt{\mathrm{tg}^2 x + 1}}\right| = \ln|\cos x \cdot (\mathrm{tg}\,x + 1)|$$

$$x \neq -\frac{\pi}{4} \ (\mathrm{tg}\,x \neq -1). \qquad = \ln|\cos x + \sin x| \;;$$

$F_0'(t) = f(t)$; platí dokonce i pro $x = \pm \frac{\pi}{2}$.

pokud spojitě rozšíříme: $\quad f(t) = \dfrac{\cos x - \sin x}{\cos x + \sin x}$

(44) $$\int \frac{1+\mathrm{tg}^2 x}{2+\mathrm{tg}\,x}\,dx = \left|\; \begin{matrix} t = \mathrm{tg}\,x \\ dx = \dfrac{dt}{1+t^2} \end{matrix} \right\} = \int \frac{1+t^2}{2+t} \cdot \frac{dt}{1+t^2} = \int \frac{dt}{t+2}$$

$$= \ln|t+2| = \ln|\mathrm{tg}\,x + 2| \;; \quad x \in \left(-\frac{\pi}{2}, -\mathrm{arctg}\,2\right),\ \left(-\mathrm{arctg}\,2, \frac{\pi}{2}\right)$$

a dále π-periodicky.