Pravidla pro „malé o": $(x \to 0)$

1. $f(x) = o(x^n)$, $g(x) = o(x^m)$, $m \geq n \Rightarrow f(x) + g(x) = o(x^n)$

2. $f(x) = o(x^n)$, $g(x) = o(x^m) \Rightarrow f(x)\,g(x) = o(x^{m+n})$

3. $f(x) = o(x^n) \Rightarrow x^m f(x) = o(x^{n+m})$

$$\frac{f(x)}{x^m} = o(x^{n-m})$$

4. $f(x) = o(x^n)$, $g(x) \sim x^m$; $m > 0 \Rightarrow f(g(x)) = o(x^{mn})$

speciálně: $f(x) = o(x^n)$

$g(x) = x^m + o(x^m)$

$$\Rightarrow f(g(x)) = o(x^{mn})$$

"Taylor"

el. fce: $e^x = 1 + x + \frac{x^2}{2!} + \dots \frac{x^n}{n!} + o(x^n)$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2})$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n+1})$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots (-1)^{n+1} \frac{x^n}{n} + o(x^n)$$

$$(1+x)^a = 1 + ax + \frac{a\cdot(a-1)}{2} x^2 + \dots \binom{a}{k} x^k + o(x^k).$$

$\boxed{f(x) = o(x^n)}$ pro $x \to 0$: $\lim\limits_{x\to 0} \frac{f(x)}{x^n} = 0$.

(tj. $f(x)$ je mnohem menší než x^n pro $x \to 0$).

$$\binom{a}{k} = \frac{a\cdot(a-1)\cdots(a-k+1)}{k!}$$

$$T^f_{x_0,n} := \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

(vždy $x_0 = 0$).

Př.: $\lim\limits_{x\to 0} \dfrac{\ln(1+xe^x)\cdot \sin(2x^2)}{x^3} \overset{!}{=} 2$

$$\sin(2x^2) = (2x^2) - \frac{1}{6}(2x^2)^3 + \underbrace{o\left((2x^2)^4\right)}_{=o(x^8).}$$

$$= 2x^2 - \frac{4}{3}x^6 + o(x^8).$$

$$\ln(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} + \dots$$

$$\ln(1+xe^x) = xe^x - \frac{1}{2}x^2e^{2x} + o\left((xe^x)^2\right)$$

$$xe^x = x\left(1+x+\frac{x^2}{2}+o(x^2)\right) = x+x^2+\frac{x^3}{2}+o(x^3)$$

$$\cancel{\frac{(xe^x)^3}{3!}} \cancel{= x + x^2 + o(x^2).} = \frac{x^3}{6} + o(x^3)$$

$$x^2e^{2x} = x^2\left(1+2x+o(x^2)\right)$$

$$= x^2+2x^3+o(x^3) \qquad + \frac{1}{6}x^3 \cancel{+o}(x^3)$$

~~AH~~

$$\ln(1+xe^x) = x + x^2 + \frac{x^3}{2} - \frac{x^2}{2} - x^3 + o(x^3)$$

$$= x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + o(x^3)$$

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celkem: $\ln(1+xe^x)\cdot\sin(2x^2)$

$$= \left(x+\frac{1}{2}x^2+o(x^2)\right)\cdot\left(2x^2-\frac{4}{3}x^6+o(x^8)\right)$$

$$= 2x^3 + x^4 + o(x^4)$$

mx

Př.: $\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$; $\longrightarrow \exp\left(-\frac{1}{6}\right)$; $x \to 0$

$$\frac{1}{x^2} \ln\left(\frac{\sin x}{x}\right)$$

(aproximace !!)

$$\frac{\sin x}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} + o(x^4)$$

$$\ln(1+y) = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 + o(y^3)$$

$$\ln\left(\frac{\sin x}{x}\right) = \ln\Big(\cancel{\ \ }$$

$$y = -\frac{x^2}{6} + \frac{x^4}{120} + o(x^4).$$

$$-\frac{x^2}{6} + \frac{x^4}{120} + o(x^4) - \frac{1}{2}\left(-\frac{x^2}{6} + \frac{x^4}{120} + o(x^4)\right)^2$$

do x^4:

$$\frac{1}{36}x^4 + (2)\left(-\frac{1}{6}\right)\left(\frac{1}{120}\right)x^6 + \left(\frac{1}{120}\right)^2 x^8 + \dots$$

$$= \frac{1}{36}x^4 + o(x^4)$$

$$= -\frac{1}{6}x^2 + \left(\frac{1}{120} - \frac{1}{72}\right)x^4 + o(x^4)$$

$$= -\frac{1}{6}x^2 - \frac{1}{180}x^4 + o(x^4).$$

$$\frac{1}{x^2}\ln\left(\frac{\sin x}{x}\right) = -\frac{1}{6} - \frac{x^2}{180} + o(x^2) \longrightarrow -\frac{1}{6}$$

mx

Pří:

$$\frac{1}{\cos x + \sin x} = \quad ; \qquad x_0 = 0.$$

$$\cos x + \sin x = 1 + x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + o(x^3)$$

$$\frac{1}{\cos x + \sin x} = f(x) = A + Bx + Cx^2 + o(x^2)$$

$$1 = \left(1 + x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + o(x^3)\right) \cdot \left(A + Bx + Cx^2 + o(x^2)\right)$$

x^0: $1 = A$

x^1: $0 = B + A$: $B = -1$

x^2: $0 = C + B - \frac{1}{2}A$: $C = -B + \frac{1}{2}A$

$= 3/2$

$$\frac{1}{\cos x + \sin x} = 1 - x + \frac{3}{2}x^2 + o(x^2).$$

mx

1.) $\dfrac{\cosh x - \sqrt{\cos x}}{x^2}$; $x \to 0$.

$$\cosh x = 1 + \frac{x^2}{2} + o(x^2)$$

$$\cos x = 1 - \frac{1}{2}x^2 + o(x^2)$$

$$\sqrt{1+y} = 1 + \frac{1}{2}y + o(y)$$

$$\sqrt{(\cos x)} = \left(1 + \left(-\frac{1}{2}x^2 + o(x^2)\right)\right)^{\frac{1}{2}}$$

$$= 1 + \frac{1}{2}\left(-\frac{1}{2}x^2 + o(x^2)\right) + o\left(-\frac{1}{2}x^2 + o(x^2)\right)$$

$$= 1 - \frac{1}{4}x^2 + o(x^2).$$

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ezzel: $\cancel{1} + \dfrac{x^2}{2} + o(x^2) - \left(\cancel{1} - \dfrac{1}{4}x^2 + o(x^2)\right)$

$$= \left(\frac{1}{2} + \frac{1}{4}\right)x^2 + o(x^2).$$

$$f(x) = \frac{3/4\, x^2 + o(x^2)}{x^2} = \frac{3}{4} + o(1) \to \frac{3}{4}.$$

mx

(2.) $\operatorname{tg} x = ?$ $(n=5)$

lichá fce: $\operatorname{tg} x = Ax + Bx^3 + Cx^5 + o(x^5)$

$\sin x = \operatorname{tg} x \cdot \cos x$

$$x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5) = \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5)\right) \cdot$$

$$\cdot \left(Ax + Bx^3 + Cx^5 + o(x^5)\right)$$

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násobíti:

x^1: $1 = 1 \cdot A$: $A = 1$

x^3: $-\frac{1}{6} = B - \frac{A}{2}$; $B = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$

x^5: $\frac{1}{120} = C - \frac{1}{2}B + \frac{1}{24}A$

$$C = \frac{1}{120} + \frac{1}{6} - \frac{1}{24} = \frac{1+20-5}{120} = \frac{16}{120} = \frac{2}{15}$$

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výsledek: $\operatorname{tg} x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + o(x^6)$.

$$\frac{\operatorname{tg} x - x}{x - \sin x} = \frac{x + \frac{1}{3}x^3 + o(x^3) - x}{x - \left(x - \frac{x^3}{6} + o(x^3)\right)} = \frac{\frac{1}{3}x^3 + o(x^3)}{+\frac{1}{6}x^3 + o(x^3)}$$

$$= \frac{\frac{1}{3} + o(1)}{\frac{1}{6} + o(1)} \to \frac{6}{3} = 2.$$

(3.) $e^{\sin x} = ?$ $\quad (n=3)$

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)$$

$$e^{\sin x} = 1 + \sin x + \frac{1}{2}\sin^2 x + \frac{1}{6}\sin^3 x + o(\sin^3 x)$$

$$\sin x = x - \frac{x^3}{6} + o(x^3)$$

$$\sin^2 x = \left(x - \frac{x^3}{6} + o(x^3)\right)\cdot\left(x - \frac{x^3}{6} + o(x^3)\right)$$

$$= x^2 + 2x\left(-\frac{x^3}{6}\right) + o(x^4)$$

$$= x^2 - \frac{1}{3}x^4 + o(x^4) =$$

$$\sin^3 x = x^3 + o(x^3)$$

$$e^{\sin x} = 1 + x + \frac{x^2}{2} + \underbrace{\left(-\frac{1}{6} + \frac{1}{6}\right)}_{0} x^3 + o(x^3)$$

$$= 1 + x + \frac{1}{2}x^2 + o(x^3).$$

mx

4. $\lim\limits_{x\to 0} \left(\frac{(1+x)\ln(1+x)}{x^2} - \frac{1}{x} \right)$

$$f(x) = \frac{1}{x^2} \cdot \left((1+x)\ln(1+x) - x \right)$$

$$\ln(1+x) = x - \frac{x^2}{2} + o(x^2)$$

$$(1+x)\ln(1+x) = (1+x)\left(x - \frac{x^2}{2} + o(x^2)\right)$$

$$= x - \frac{x^2}{2} + x^2 + o(x^2)$$

$$= x + \frac{1}{2}x^2 + o(x^2)$$

$$f(x) = \frac{\frac{1}{2}x^2 + o(x^2)}{x^2} = \frac{1}{2} + o(1) \to \frac{1}{2}$$

mx

⑤ $$\lim_{x\to 0} \frac{\cos(xe^{x})-\cos(xe^{-x})}{x^3} = -2$$

⑥ $$\lim_{x\to 0} \frac{(\cosh x^2-1)\ln(\cos x)}{x^6} = -\frac{1}{4}$$

$$\cos x = 1-\frac{x^2}{2}+o(x^3)$$

$$xe^{x} = x\left(1+x+\frac{x^2}{2}+o(x^2)\right) = x+x^2+\frac{x^3}{2}+o(x^3)$$

$$xe^{-x} = x\left(1-x+\frac{x^2}{2}+o(x^2)\right) = x-x^2+\frac{x^3}{2}+o(x^3).$$

$$\cos(xe^{x}) = 1-\frac{1}{2}\left(x+x^2+\frac{x^3}{2}+o(x^3)\right)^2 + o(\ldots)$$

$$+\, o\left(\left(x+o(x)\right)^3\right)$$

$$= 1-\frac{1}{2}\left(x^2+2x^3+o(x^3)\right)+o(x^3)$$

$$= 1-\frac{1}{2}x^2-x^3+o(x^3).$$

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analogicky: $\cos(xe^{-x}) = 1-\frac{1}{2}x^2+x^3+o(x^3)$

⑤ $$f(x) = \frac{1}{x^3}\left(-2x^3+o(x^3)\right)$$

$$= -2+o(1) \to -2,\ x\to 0.$$

mx

⑥ $\cosh x = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \cdots \frac{x^{2n}}{2n!} + o(x^{2n})$

$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+1})$

$$\cosh x^2 - 1 = \frac{x^4}{2} + \frac{x^8}{24} + o(x^8)$$

$$\cos x = 1 - \frac{x^2}{2} + o(x^2)$$

$$\ln \cos x = \ln\Big(1 \underbrace{- \frac{x^2}{2} + o(x^2)}_{y}\Big)$$

$$= y + o(y) = -\frac{x^2}{2} + o(x^2) + o\left(-\frac{x^2}{2} + o(x^2)\right)$$

$$= -\frac{x^2}{2} + o(x^2).$$

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celkem: $\left(\frac{x^4}{2} + \frac{x^8}{24} + o(x^8)\right)\left(-\frac{x^2}{2} + o(x^2)\right)$

$$= -\frac{1}{4}x^6 + \underbrace{\frac{1}{2}x^4 o(x^2) - \frac{1}{48}x^{10} + \frac{1}{24}x^8 o(x^2) + \cdots}_{o(x^6)}$$

$$\lim_{x\to 0} \frac{(\cosh x^2 - 1)\ln\cos x}{x^6} = \lim_{x \to 0} \frac{-\frac{1}{4}x^6 + o(x^6)}{x^6}$$

$$= -\frac{1}{4}.$$

nx