for the Hausdorff semidistance between A and B. Note that if B is closed then $\text{dist}(A, B) = 0$ implies that $A \subseteq B$.

Theorem 1.5 *Let* $I_2 = [-\frac{1}{2}, \frac{1}{2}]^2 \subset \mathbb{R}^2$. *Then* $\dim(I_2) \geq 2$.

Proof We want to show that any covering α of I_2 with sufficiently small mesh size contains at least three sets with nonempty intersection. To this end, take a covering α with mesh size < 1 so that no element of the covering contains points of opposite faces.

The first step is to construct a refinement $\tilde{\alpha}$ of α consisting of closed, rather than open, sets. To do this, observe that every $x \in I_2$ is contained in some $U_x \in \alpha$, and we can find an open set V_x such that $x \in V_x \subset \bar{V}_x \subset U_x$. Since I_2 is compact and $\{V_x : x \in I_2\}$ is an open cover of I_2, there is a finite subcover $\{V_{x_j}\}$. We take $\tilde{\alpha}$ to be the collection of all the closed sets $\{\bar{V}_{x_j}\}$. By construction this is a refinement of α consisting of closed sets.

We now show that $\tilde{\alpha}$ contains at least three sets with nonempty intersection, from which it is immediate (since $\tilde{\alpha}$ is a refinement of α) that α contains at least three sets with nonempty intersection.

Let Γ_1 denote the side of I_2 with $x = -\frac{1}{2}$, Γ_1' the side with $x = \frac{1}{2}$, Γ_2 the side with $y = -\frac{1}{2}$, and Γ_2' the side with $y = \frac{1}{2}$. Let L_1 denote the union of those elements of $\tilde{\alpha}$ that intersect Γ_1; L_2 the union of those elements of $\tilde{\alpha}$ that are not in L_1 and intersect Γ_2; and let L_3 be the union of all the other elements of $\tilde{\alpha}$ (those that intersect neither Γ_1 nor Γ_2). See Figure 1.1(a).

If we define $K_1 = L_1 \cap L_3$ then K_1 separates Γ_1 and Γ_1' in I_2, i.e. there exist open sets U_1 and U_1' such

$$I_2 \setminus K_1 = U_1 \cup U_1', \qquad U_1 \cap U_1' = \emptyset$$

and $\Gamma_1 \subset U_1$, $\Gamma_1' \subset U_1'$. The set $K_2' = L_1 \cap L_2 \cap L_3$ separates $\Gamma_2 \cap K_1$ from $\Gamma_2' \cap K_1$ in K_1. One can then find a new closed set K_2, with $K_2 \cap K_1 \subseteq K_2'$, that separates Γ_2 and Γ_2' in I_2, i.e. such that there exist open sets U_2 and U_2' such that

$$I_2 \setminus K_2 = U_2 \cup U_2', \qquad U_2 \cap U_2' = \emptyset$$

and $\Gamma_2 \subset U_2$, $\Gamma_2' \subset U_2'$. These constructions are illustrated in Figure 1.1(b). (If the 'proof by diagram' of this last step is unconvincing, see IV.3 A) in Hurewicz & Wallman (1941), or Exercise 1.3.)

Now for each $x \in I_2$, let $v(x)$ be the 2-vector with components

$$v_i(x) = \begin{cases} \text{dist}(x, K_i) & x \in U_i, \\ 0 & x \in K_i, \\ -\text{dist}(x, K_i) & x \in U_i', \end{cases}$$

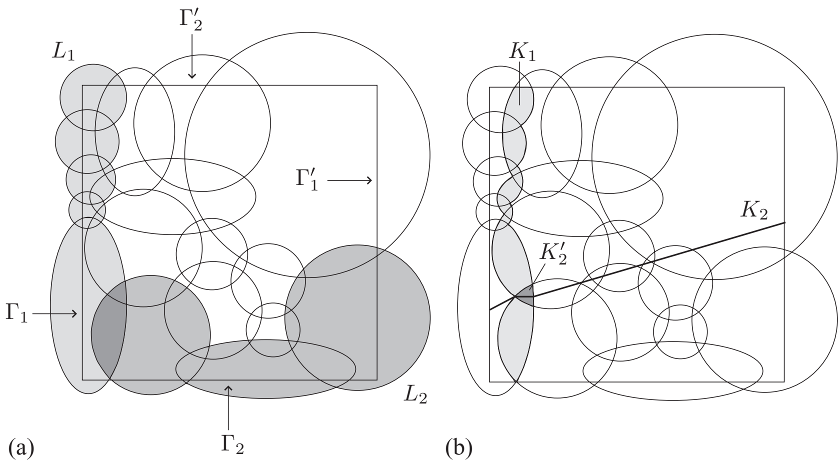

Figure 1.1 (a) A covering of I_2, divided into sets L_1 (lightly shaded), L_2 (more heavily shaded), and L_3 (not shaded). (b) K_1 (lightly shaded) separates Γ_1 and Γ_1' in I_2; K_2' (a subset of K_1, shaded more heavily) separates $K_1 \cap \Gamma_2$ and $K_1 \cap \Gamma_2'$ in K_1; K_2 (the dark line) separates Γ_2 and Γ_2' in I_2, with $K_2 \cap K_1 \subseteq K_2'$.

and set $f(x) = x + v(x)$; note that $f(x) \in I_2$, and that f is continuous. It follows from the Brouwer Fixed Point Theorem (Theorem 1.4) that f has a fixed point, i.e. there exists an $x_0 \in I_2$ such that $f(x_0) = x_0$. In particular, this implies that $\mathrm{dist}(x_0, K_1) = \mathrm{dist}(x_0, K_2) = 0$, i.e. that $K_1 \cap K_2 \subset K_2' = L_1 \cap L_2 \cap L_3$ is nonempty. Since each of the original elements of $\tilde{\alpha}$ is contained in only one of the L_js, there are three elements of $\tilde{\alpha}$ that contain a common point. □

1.3 Embedding sets with finite covering dimension

We now prove the fundamental embedding result that any space with covering dimension n can be topologically embedded into $\mathbb{R}^{2n+1}$; note that this characterises sets of finite covering dimension as homeomorphic images of subsets of finite-dimensional Euclidean spaces. The embedding result in the compact case (which we treat here) is due to Menger (1926) and Nöbeling (1931); we follow the presentation of Hurewicz & Wallman (1941, Theorem V.2) and Munkres (2000, Theorem 50.5). A similar result is possible in the general (non compact) case, see Theorem V.3 in Hurewicz & Wallman (1941).

The proof uses the Baire Category Theorem, which we state here for convenience. For a proof see Munkres (2000, Theorem 48.2), for example.