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Finite-dimensional limiting dynamics for dissipative parabolic equations

A. V. Romanov

Abstract. For a broad class of semilinear parabolic equations with compact attractor $\mathcal{A}$ in a Banach space E the problem of a description of the limiting phase dynamics (the dynamics on $\mathcal{A}$) of a corresponding system of ordinary differential equations in $\mathbb{R}^N$ is solved in purely topological terms. It is established that the limiting dynamics for a parabolic equation is finite-dimensional if and only if its attractor can be embedded in a sufficiently smooth finite-dimensional submanifold $\mathcal{M} \subset E$. Some other criteria are obtained for the finite dimensionality of the limiting dynamics:

a) the vector field of the equation satisfies a Lipschitz condition on $\mathcal{A}$;
b) the phase semiflow extends on $\mathcal{A}$ to a Lipschitz flow;
c) the attractor $\mathcal{A}$ has a finite-dimensional Lipschitz Cartesian structure.

It is also shown that the vector field of a semilinear parabolic equation is always Hölder on the attractor.

Bibliography: 19 titles.

Introduction

Many non-stationary problems in mathematical physics can be written as a dissipative semilinear parabolic equation

$$\partial_t u = -Au + F(u) \tag{1}$$

in a Banach space E with norm $|\cdot|$. Here $u(t) \in E$, $\partial_t u$ is the strong derivative with respect to t, and A is a linear sectorial operator acting in E with dense domain $D(A)$, compact resolvent, and spectrum in the right half-plane. We assume that for some $\alpha \in [0,1)$ the non-linear function F is in $C^2(E^\alpha, E)$, where $E^\alpha = D(A^\alpha)$ has norm $|u|_\alpha = |A^\alpha u|$. It is also assumed that equation (1) generates in E^α a semiflow $\{\Phi_t\}_{t\geqslant 0}$ of class C^2 with a compact attractor $\mathcal{A}$: a maximal bounded invariant set that uniformly attracts balls in E^α as $t \to +\infty$.

The idea of the finite-dimensional behaviour of solutions of parabolic equations for large time goes back to the work of Hopf [1]. The contemporary understanding of this amounts to the equivalence in some sense or another of the asymptotic behaviour (as $t \to +\infty$) of the dynamics of equation (1) and of some system of

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ordinary differential equations (ODEs) in $\mathbb{R}^N$. Under our conditions the attractor $\mathcal{A}$ has finite fractal dimension, as follows immediately from the well-known result of Mane [2] on invariant sets of smooth compact maps acting in Banach spaces. However, the structure of $\mathcal{A}$ can be very irregular, and thus finite dimensionality alone of the attractor of equation (1) is not yet enough to determine an ODE with analogous dynamics as $t \to +\infty$.

More radical approaches to the problem of a finite-dimensional description of the asymptotic dynamics of parabolic equations involve the concept of an inertial manifold: a smooth finite-dimensional invariant surface $\mathcal{M} \subset E^\alpha$ containing the attractor and exponentially attracting all the solutions $u(t)$ as $t \to +\infty$. The existence of such a manifold (with a suitable Cartesian structure) enables us to easily construct an inertial form: an ODE in $\mathbb{R}^N$ completely describing the behaviour of the solutions of the original problem for large time. The theory of inertial manifolds has fairly stringent requirements on the spectral properties of the linear part of (1) (see, for example, [3]–[5]), and this restricts its area of application.

In this article we discuss a characteristic of evolution equations that is intermediate between finite dimensionality of the attractor and existence of an inertial manifold. We say that the equation (1) has *finite-dimensional limiting dynamics* if there is an ODE

$$\dot{x} = h(x) \tag{2}$$

with a Lipschitz vector field $h(x)$ and phase flow $\{S_t\}_{t\in\mathbb{R}}$ in $\mathbb{R}^N$, and also a compact set $V \subset \mathbb{R}^N$ that is invariant for $\{S_t\}$ such that the semiflow $\{\Phi_t\}$ on the attractor $\mathcal{A}$ is Lipschitz-conjugate to the semiflow $\{S_t\}_{t\geqslant 0}$ on V. It is thereby required that the limiting regimes of the original problem are 'embedded' in the limiting regimes of some ODE in $\mathbb{R}^N$. In fact, the ODE (2) describes the dynamics of equation (1) on $\mathcal{A}$. The existence of an inertial manifold implies for (1) the finite dimensionality of the limiting dynamics (the dynamics on the attractor).

Our main result (Theorem 1.5) is that the possibility of embedding the attractor $\mathcal{A}$ in a sufficiently smooth finite-dimensional submanifold $\mathcal{M} \subset E^\alpha$ ensures that the limiting dynamics of equation (1) is finite-dimensional. More precisely, the limiting dynamics is finite-dimensional if $\mathcal{A} \subset \mathcal{M}$, where $\mathcal{M}$ is a compact C^2-submanifold of finite dimension in E^α. The converse is true with the smoothness condition $\mathcal{M} \in \mathrm{Lip}$ instead of $\mathcal{M} \in C^2$ under a slight additional restriction on the operator A in (1).

Thus, at least one of the following possibilities is realized for every equation of the form (1):

a) the dynamics on $\mathcal{A}$ is finite-dimensional;
b) the attractor $\mathcal{A}$ is not contained in any finite-dimensional C^2-manifold.

The second assumption looks fairly exotic. In any case there is at present no example of a parabolic equation with such a complicated structure of the attractor. All that is known is an example [6] of a scalar reaction-diffusion equation not having a normally hyperbolic inertial manifold of class C^1 in a cube in $\mathbb{R}^4$. For semilinear hyperbolic equations the possibility of the situation b) does in fact follow from results in [7], although the specifics of problems of this type are very essential. On the other hand, it follows from recent results of Foiaş and Olson [8] that in a

Hilbert space any compact set of finite fractal dimension can be embedded in a finite-dimensional Hölder manifold.

Furthermore, we establish (Theorem 1.6) several alternative criteria for finite dimensionality of the limiting dynamics of equation (1). One is the Lipschitz condition $|G(u)-G(v)|_\alpha \leqslant K|u-v|_\alpha$ for the vector field $G=-A+F$ on the attractor $\mathcal{A}$. Another criterion is that the semiflow $\{\Phi_t\}$ on $\mathcal{A}$ must be injective and must extend to a flow that is Lipschitz in the E^α-metric. A third criterion for finite dimensionality of the limiting dynamics is the requirement that the attractor have a Lipschitz Cartesian structure on the lowest modes of the operator A. In other words, for some $a>0$ the finite-dimensional spectral projection P corresponding to the part of the spectrum of A with $\operatorname{Re}\lambda \leqslant a$ must satisfy an estimate of the form $|u-v|_\alpha \leqslant K|P(u-v)|_\alpha$ on $\mathcal{A}$, where $K=K(a)$. Yet another criterion reduces to an analogous estimate for an arbitrary finite-dimensional projection P that is bounded on E^α (and also on $E^{\alpha-1}$ if E is non-reflexive), and a last criterion reduces to the equivalence on $\mathcal{A}$ of the metrics of the spaces E^α and $E^{\alpha-1}$, where $E^{\alpha-1}$ is the completion of E in the norm $|A^{\alpha-1}u|$. We explain that the conclusions in Theorem 1.6 are valid under the same restriction as in Theorem 1.5 on the linear operator A (for example, in the Hilbert case all operators with 'leading' normal part work).

Of interest in connection with the first of the criteria for finite dimensionality of the limiting dynamics (the vector field is of Lipschitz class on $\mathcal{A}$) is Theorem 4.1, which says that for equations of the form (1) the vector field $G=-A+F$ is always Hölder on the attractor in the E^α-metric, with exponent $\beta=(1-\alpha)/(2-\alpha)$. We remark that the Navier–Stokes system with a sufficiently regular external force on a two-dimensional torus Ω admits an estimate $|G(u)-G(v)|_\alpha \leqslant K|u-v|_\alpha^\beta$ for $u,v\in\mathcal{A}$, any $\beta<1$, and a number $K=K(\beta,\mathcal{A})$. Here E is a suitable subspace of $L^2(\Omega;\mathbb{R}^2)$ and $\alpha>1/2$.

§ 1. Main results

In an infinite-dimensional separable Banach space E with norm $|\cdot|$ we consider the evolution equation (1) with a linear operator A and a non-linearity F. The closed operator A with dense domain $D(A)\subset E$ is assumed to be sectorial, that is, $(-A)$ generates an analytic semigroup $\{e^{-tA}\}_{t\geqslant 0}$ of bounded operators on E. Here and below we use facts from [9] about semilinear parabolic equations (sometimes without comment). For a sectorial operator A with spectrum $\sigma(A)$ in the half-plane $\operatorname{Re}\lambda>0$ (we shall write $\operatorname{Re}\sigma(A)>0$) it is possible to define unambiguously the powers A^θ for all $\theta\in\mathbb{R}$ along with a corresponding scale $\{E^\theta\}$ of Banach spaces by letting E^θ be $D(A^\theta)$ for $\theta\geqslant 0$ and the completion of E in the norm $|A^\theta u|$ for $\theta<0$. In both cases the spaces E^θ are equipped with the graph norm $|u|_\theta=|A^\theta u|$. We remark that if E is a Hilbert space, then the spaces E^θ are also Hilbert spaces. For $\nu<\theta$ the embedding $E^\theta\subset E^\nu$ is dense and continuous:

$$|u|_\nu \leqslant c(\nu,\theta)|u|_\theta \tag{3}$$

for $u\in E^\theta$. Suppose also that for some $0\leqslant\alpha<1$ the function $F\colon E^\alpha\to E$ is twice continuously differentiable ($F\in C^2(E^\alpha,E)$).

The above requirements on A and F ensure in general only the local (for $t>0$) solubility in E^α of equation (1) with initial condition $u(0)\in E^\alpha$. However, we shall assume that (1) is dissipative in E^α. This means the existence on E^α of a continuous semiflow $\{\Phi_t\}_{t\geqslant 0}$, $\Phi_t u_0 = u(t)$ with $u_0 = u(0)$, and the existence in E^α of a bounded set U such that $\Phi_t B \subset U$ for every ball $B \subset E^\alpha$ and for $t > \tau(B)$. It turns out that $\{\Phi_t\}$ is a semiflow of class C^2 on E^α and $\Phi_t u \in E^1$ for $u \in E^\alpha$ and $t > 0$. But if in addition the operator A is discrete (has compact resolvent) and F is bounded on balls $B \subset E^\alpha$, then for $t > \tau(B)$ the evolution operators Φ_t are compact on B.

Thus, we start with the following basic hypotheses about the coefficients and the dynamics of equation (1).

(H1) The linear operator A is sectorial and discrete, and it has a countable spectrum $\sigma(A)$ with $\operatorname{Re}\sigma(A) > 0$.

(H2) For some $\alpha \in [0, 1)$ the function F is in $C^2(E^\alpha, E)$ and is bounded on balls in E^α.

(H3) Equation (1) is dissipative in E^α.

We remark that $|\arg\lambda| < \varphi < \pi/2$ for $\lambda \in \sigma(A)$, where $\varphi = \varphi(A)$.

A set $\mathcal{N} \subset E^\alpha$ is said to be *invariant* if $\Phi_t\mathcal{N} = \mathcal{N}$ for any $t > 0$. Under the hypotheses (H1)–(H3) the semiflow $\{\Phi_t\}$ has a compact maximal invariant set: the attractor $\mathcal{A} \subset E^\alpha$ (in this connection see [4], § 2). Each invariant set $\mathcal{N}$ is contained in E^1 (in particular, $\mathcal{A} \subset E^1$), and if $\mathcal{N}$ is bounded in E^α, then $\mathcal{N} \subset \mathcal{A}$.

According to the hypothesis (H2) the function $F\colon E^\alpha \to E$ is locally Lipschitz and hence

$$|F(u) - F(v)| \leqslant L|u - v|_\alpha \tag{4}$$

on every compact set $\mathcal{K} \subset E^\alpha$, with $L = L(\mathcal{K})$.

For arbitrary metric spaces V_1 and V_2 we denote by $\operatorname{Lip}(V_1, V_2)$ the class of Lipschitz maps from V_1 to V_2. A continuous flow $\{S_t\}$ on a space V with metric ρ is said to be *Lipschitz* if $S_t \in \operatorname{Lip}(V, V)$ for all $t \in \mathbb{R}$. In this case (see [10], 8A.10)

$$\rho(S_t x, S_t y) \leqslant K e^{\omega|t|}\rho(x, y) \tag{5}$$

for $x, y \in V$ and $t \in \mathbb{R}$, with constants $K \geqslant 1$ and $\omega \geqslant 0$ independent of (x, y, t). It is well known that an ODE $\dot{x} = h(x)$ with $h \in \operatorname{Lip}(\mathbb{R}^N, \mathbb{R}^N)$ generates a Lipschitz flow in $\mathbb{R}^N$.

Definition 1.1. Suppose that $\mathcal{K}$ is a compact invariant set in E^α. We say that *the dynamics on* $\mathcal{K}$ *is finite-dimensional* if for some $N \geqslant 1$ there exist an ODE $\dot{x} = h(x)$ with Lipschitz vector field $h(x)$ in $\mathbb{R}^N$ and a phase flow $\{S_t\}$, along with a Lipschitz embedding $g\colon \mathcal{K} \to \mathbb{R}^N$, such that $g\Phi_t u = S_t g u$ for $u \in \mathcal{K}$ and $t \geqslant 0$.

Remark 1.2. It follows at once from this definition that the evolution operators Φ_t are injective on $\mathcal{K}$ for $t > 0$. Setting $\Phi_{-t} = g^{-1}S_{-t}g$, we get a Lipschitz flow $\{\Phi_t\}_{t\in\mathbb{R}}$ on $\mathcal{K}$.

Definition 1.3. If the dynamics on the attractor $\mathcal{A}$ is finite-dimensional, then we say that *the limiting dynamics of equation* (1) *is finite-dimensional.*

Let I be the identity operator on Banach spaces, and let $\|\cdot\|$ and $\|\cdot\|_\theta$ be the norms of bounded linear operators on E and E^θ, respectively. We arrange the set $\{\mu = \operatorname{Re}\lambda : \lambda \in \sigma(A)\}$ in increasing order $(0 < \mu_1 < \mu_2 < \cdots)$. Denote by

P_n the (bounded on E) spectral projection of A corresponding to the part of the spectrum with $\operatorname{Re}\lambda \leqslant \mu_n$ (see [11], § 3.2.2) and let $Q_n = I - P_n$. The projection P_n is finite-dimensional (of rank $k \geqslant n$) and $P_nA = AP_n$ on $D(A) = E^1$. The finite-dimensional subspace P_nE is generated by eigenvectors and associated (root) vectors of A, and thus $P_nE = P_nE^\theta \subset E^\theta$ and P_n, Q_n are bounded projections on E^θ for any $\theta > 0$ (see (3)). It is important to note that P_n and Q_n commute with the operators e^{-tA} for $t > 0$. This follows, for example, from the standard representation of e^{-tA} as the strong limit as $m \to \infty$ of the bounded linear operators $(I + m^{-1}tA)^{-m}$ on E.

The following estimates are known ([9], § 1.5):

$$|e^{-tA}u|_\alpha \leqslant M(n,\alpha,\omega_n)t^{-\alpha}e^{-\omega_n t}|u| \tag{6}$$

for $n \geqslant 1$, $u \in Q_nE$, and arbitrary $\omega_n \in (0, \mu_{n+1})$. In some of the statements below it must be assumed that the linear operator A satisfies the following additional requirement.

Hypothesis 1.4. There is a sequence of positive numbers $\omega_n < \mu_{n+1}$ such that $\omega_n \to \infty$ as $n \to \infty$ and

$$\inf_{n\geqslant 1} \omega_n^{\alpha-1} M(n,\alpha,\omega_n)\|Q_n\| = 0, \tag{7}$$

where $M(n,\alpha,\omega_n)$ is the minimal constant in (6).

If E is a Hilbert space and the operator A is self-adjoint, then $\|Q_n\| = 1$ and a straightforward computation gives us that $M(n,\alpha,\omega_n) \leqslant (\alpha\mu_{n+1}/(e\varepsilon))^\alpha$ with $\varepsilon = \mu_{n+1} - \omega_n$, so that Hypothesis 1.4 is satisfied, for example, with $\omega_n = \mu_{n+1}/2$. The same is true for a normal operator A. In the general case, however, the quantities $\|Q_n\|$ and $M(n,\alpha,\omega_n)$ can have any order of growth with respect to n and ω_n, depending on the spectral properties of A.

Our main result is the following.

Theorem 1.5. *If the attractor $\mathcal{A}$ is contained in a compact finite-dimensional C^2-submanifold $\mathcal{M}$ of E^α, then the limiting dynamics of equation* (1) *is finite-dimensional. Conversely, if the limiting dynamics of* (1) *is finite-dimensional, the operator A satisfies Hypothesis* 1.4, *and the P_n denote its spectral projections, then the attractor $\mathcal{A}$ is part of a finite-dimensional Lipschitz manifold $\mathcal{M}$ in E^α that is the graph of a function $f \in \operatorname{Lip}(P_nE^\alpha, Q_nE^\alpha)$ for some $n \geqslant 1$ and for $Q_n = I - P_n$.*

The conclusions of the theorem are valid when $\mathcal{A}$ is replaced by any compact invariant set $\mathcal{K} \subset E^\alpha$ if we are speaking of the "finite-dimensional dynamics on $\mathcal{K}$" in the sense of Definition 1.1.

Let $G(u) = -Au + F(u)$ be the vector field of equation (1). If $\mathcal{N}$ is an invariant subset of E^α, then $G(u) \in E^\theta$ for $u \in \mathcal{N}$ for any $\theta < 1$ (see [9], Theorem 3.5.2).

We now formulate a statement giving several different necessary and sufficient conditions for finite dimensionality of the dynamics of (1) on an arbitrary compact invariant set $\mathcal{K} \subset \mathcal{A}$ and, in particular, on the attractor $\mathcal{A}$.

Theorem 1.6. *Suppose that the operator A satisfies Hypothesis* 1.4 *and $\mathcal{K}$ is a compact invariant subset of E^α. Then the following assertions are equivalent.*

(FD) *The dynamics on $\mathcal{K}$ is finite-dimensional.*

(VF) $|G(u)-G(v)|_\alpha \leqslant C|u-v|_\alpha$ *on $\mathcal{K}$, where $C=C(\mathcal{K})$.*

(Fl) *The semiflow $\{\Phi_t\}$ on $\mathcal{K}$ is injective and extends to a flow that is Lipschitz in the E^α-metric.*

(GrF) *For some $n \geqslant 1$ the spectral projection P_n satisfies the estimate $|u-v|_\alpha \leqslant C|P_n(u-v)|_\alpha$ on $\mathcal{K}$, where $C=C(\mathcal{K},n)$.*

(Gr) *There is a finite-dimensional projection P that is bounded on E^α* (*and on $E^{\alpha-1}$ if E is non-reflexive*) *and such that $|u-v|_\alpha \leqslant C|P(u-v)|_\alpha$ on $\mathcal{K}$, where $C=C(\mathcal{K},P)$.*

(EM) *The metrics of E^α and $E^{\alpha-1}$ are equivalent on $\mathcal{K}$.*

When $\mathcal{K}=\mathcal{A}$ the theorem gives five different criteria for finite dimensionality of the limiting dynamics of equation (1). The condition (VF) means that the vector field G is Lipschitz on $\mathcal{K}$. The condition (EM) reduces (see (3)) to the estimate $|A(u-v)|_{\alpha-1} \leqslant C|u-v|_{\alpha-1}$ on $\mathcal{K}$. According to the condition (GrF) the map $P_n\colon \mathcal{K} \to P_nE^\alpha$ is a Lipschitz embedding, which implies that $\mathcal{K}$ is part of a Lipschitz graph over P_nE^α. The meaning of the condition (Gr) is the same, but with the spectral projection P_n replaced by a finite-dimensional projection of general form.

Remark 1.7. The closed logical cycle (VF)→(Fl)→(GrF)→(Gr)→(EM)→(VF) is in fact established for Theorem 1.6, along with (FD)→(Fl) and (Gr)+(VF)→(FD). We point out that the restriction 1.4 on the operator A is used here only in the implication (Fl)→(GrF).

In connection with the necessity of verifying Hypothesis 1.4 we note that for the given equation (1) the representation of the vector field as the sum of the linear part $(-A)$ and the non-linearity F is not unique. Let $A_0=A-TA^\alpha$, where T is a bounded linear operator on E, and suppose that the spectrum $\sigma(A_0)$ is countable and $\operatorname{Re}\sigma(A_0)>0$. From results in [9] (§ 1.4) it follows that the operator A_0 with domain $D(A_0)=D(A)$ is sectorial and discrete. Moreover, for $0 \leqslant \theta \leqslant 1$ the spaces E^θ and $E_0^\theta=D(A_0^\theta)$ coincide, and the graph norms in them are equivalent. From this we see that equation (1), written in the form $\partial_t u=-A_0u+F_0(u)$ with $F_0=F-TA^\alpha$, satisfies the main hypotheses (H1)–(H3), and it is possible that A_0 satisfies Hypothesis 1.4 but A does not. Thus, we can say that in Theorems 1.5 and 1.6 the linear operator A must satisfy Hypothesis 1.4 up to perturbations of the form TA^α with the operator T bounded on E.

§ 2. Proof of Theorem 1.5

Here we derive the main theorem from Theorem 1.6 (to be proved later).

For $\theta \in \mathbb{R}$ and $k \geqslant 1$ we denote by $\Pi(\theta,k)$ the set of projections of rank k that are bounded on E^θ. Introduction of the metric $\rho(P,Q)=\|P-Q\|_\theta$ turns $\Pi(\theta,k)$ into a complete metric space. The following assertion is a special case of Lemma 9.2.1 in [9], which is an infinite-dimensional variant of the Whitney embedding theorem.

Lemma 2.1. *Let $\theta \in \mathbb{R}$ and suppose that $\mathcal{M}$ is a compact C^2-submanifold of finite dimension m in E^θ. Then there is a projection $P \in \Pi(\theta, k)$, $k = 2m+1$, such that the restriction $P\big|_{\mathcal{M}}$ is a C^2-diffeomorphism of the manifold $\mathcal{M}$ onto $P\mathcal{M} \subset PE^\theta$.*

We need finite-dimensional projections that are simultaneously bounded in different spaces E^θ, and in this connection we must take into consideration the dual spaces $Y_\theta = (E^\theta)'$.

If the original space E is reflexive, then the adjoint A' of the sectorial operator A defined in E is densely defined and sectorial in $X = E'$ ([9], § 7.3), and this enables us to define a scale $\{X^\theta\}$ of Banach spaces corresponding to A'. For any $\theta \in \mathbb{R}$ the maps $A^\theta \colon E^\theta \to E$ and $(A')^\theta \colon X \to X^{-\theta}$ are isometries, and the same is true for the adjoint map $(A^\theta)' \colon X \to Y_\theta$. It follows from elementary properties of sectorial operators (see [9], Ch. 1) that $(A^\theta)' = (A')^\theta$, and hence $Y_\theta = X^{-\theta}$ and the embedding $Y_\nu \subset Y_\theta$ is continuous and dense for $\theta > \nu$.

Suppose now that E is an arbitrary Banach space. Since the embedding $E^\theta \subset E^\nu$ is continuous and dense for $\theta > \nu$, it follows that the embedding $Y_\nu \subset Y_\theta$ is continuous. Furthermore, the set Y_ν of functionals is total, that is, if $\zeta(u) = 0$ for all $\zeta \in Y_\nu$ and some $u \in E^\theta$, then $u = 0$. But this implies ([11], § 1.4.5) that Y_ν is weak-$*$ dense in Y_θ for $\theta > \nu$.

We show that there are 'sufficiently many' finite-dimensional projections that are bounded simultaneously in the different spaces E^θ.

Lemma 2.2. *For $\theta > \nu$ and $k \geqslant 1$ the set $\Pi(\theta, k) \cap \Pi(\nu, k)$ of projections is strongly dense in $\Pi(\theta, k)$ (norm dense if E is reflexive).*

Proof. If $P \in \Pi(\theta, k)$, then

$$Pu = \sum_{i=1}^{k} \zeta_i(u) u_i \tag{8}$$

for $u \in E^\theta$, where $\{u_i\}$ and $\{\zeta_i\}$ are linearly independent mutually conjugate ($\zeta_i(u_j) = \delta_{ij}$) systems of k vectors in the spaces E^θ and Y_θ, respectively. In fact, for every projection P of the form (8) it is necessary to construct a sequence of projections of the same form on E^θ with (χ_i, v_i) instead of (ζ_i, u_i), where $\chi_i \in Y_\nu$, $\chi_i \to \zeta_i$ in the weak-$*$ topology in Y_θ, and $v_i \to u_i$ strongly in E^θ. If for some k functionals $\chi_i \in Y_\nu$ the matrix $W = \{a_{ij}\} = \{\chi_j(u_i)\}$ is invertible, then the system $\{v_i\} \subset E^\theta$ of vectors with $v_i = Tu_i$ will be conjugate to $\{\chi_i\}$, where T is the linear operator defined in the k-dimensional subspace PE^θ with matrix W^{-1} in the basis $\{u_i\}$. Using the weak-$*$ denseness of Y_ν in Y_θ we choose functionals $\chi_i \in Y_\nu$ that are weak-$*$ convergent to ζ_i in Y_θ. Then the corresponding $k \times k$ matrices W converge to I, and hence the operators T converge to I in PE^θ. Therefore, $v_i = Tu_i \to u_i$ strongly in E^θ, as required.

But if the space E is reflexive, then Y_ν is strongly dense in Y_θ and there are functionals $\chi_i \in Y_\nu$ that converge strongly to ζ_i in Y_θ, which implies the uniform convergence of the projections constructed according to the same scheme. The lemma is proved.

Lemma 2.3. *Suppose that $\theta > \nu$, $k \geqslant 1$, and $\mathcal{M}$ is a compact C^1-submanifold of finite dimension in E^θ. Then for any projection $P \in \Pi(\theta, k)$ there is a sequence of* *projections* $P_l \in \Pi(\nu, k)$ *that converge strongly* (*in* $\Pi(\theta, k)$) *to* P *and are such that* $|(P - P_l)(u - v)|_\theta \leqslant \varepsilon_l |u - v|_\theta$ *on* $\mathcal{M}$ *and* $\varepsilon_l = o(1)$ *as* $l \to \infty$.

Proof. Lemma 2.2 ensures the existence of the necessary projections P_l, but without an estimate on $\mathcal{M}$. The norms $\|P_l\|_\theta$ are uniformly bounded. Therefore, on every compact set $\mathcal{K} \subset E^\theta$ the functions $P_l \colon \mathcal{K} \to E^\theta$ are equicontinuous, and then the Arzelà–Ascoli theorem gives us that $P_l \to P$ uniformly on $\mathcal{K}$. We set $\psi(u, v) = (u - v)/|u - v|_\theta$ for $u, v \in E^\theta$ $(u \neq v)$. Everything will be proved if we establish that the values $\psi(u, v)$ for $u, v \in \mathcal{M}$ form a relatively compact set in E^θ.

Let $\{u_l\}$ and $\{v_l\}$ be any two sequences of elements of $\mathcal{M}$ with $u_l \neq v_l$. Since the manifold $\mathcal{M}$ is compact, it can be assumed that $u_l \to u$ and $v_l \to v$ as $l \to \infty$. If $u \neq v$, then $\psi(u_l, v_l) \to \psi(u, v)$ in E^θ. Therefore, we consider the case $u = v$. Let Q denote a projection that is bounded from E^θ onto the tangent space of the manifold $\mathcal{M}$ at $u \in \mathcal{M}$. We remark that $w_l = \psi(Qu_l, Qv_l) \in QE^\theta$ and, in view of the finite dimensionality of QE^θ, we can assume that $w_l \to w$ in E^θ. Further, we set $\varkappa_l = |Qu_l - Qv_l|_\theta / |u_l - v_l|_\theta$. It follows from the definition of the tangent space of a smooth manifold that $\varkappa_l \to 1$ and $u_l - v_l = Qu_l - Qv_l + |u_l - v_l|_\theta z_l$, with $|z_l|_\theta = o(1)$ as $l \to \infty$. From this we see that $\psi(u_l, v_l) = \varkappa_l w_l + z_l$. Hence, $\psi(u_l, v_l) \to w$ in E^θ as $l \to \infty$ and the lemma is proved.

For convenience of reference we formulate a vector version of the well-known theorem on extension of Lipschitz functions.

Lemma 2.4. *Suppose that* $f_0 \in \mathrm{Lip}(V, X)$, *where* V *is a compact subset of* $\mathbb{R}^N$, *and let* X *be a Banach space. Then there is a function* $f \in \mathrm{Lip}(\mathbb{R}^N, X)$ *such that* $f = f_0$ *on* V.

The lemma can be proved in the same way as in the scalar case $X = \mathbb{R}^1$ (see [12], Ch. 6, Theorem 3).

We can now proceed directly to the proof of Theorem 1.5.

Proof of Theorem 1.5. Suppose that the attractor $\mathcal{A}$ is contained in a compact C^2-manifold $\mathcal{M}$ of dimension m in E^α. By Lemma 2.1, there is a projection P of rank $k = 2m + 1$ that is bounded on E^α and such that $|u - v|_\alpha \leqslant C|P(u - v)|_\alpha$ on $\mathcal{M}$. By Lemma 2.3 (with $\theta = \alpha$, $\nu = \alpha - 1$), there is a rank-k projection P_0 that is bounded on E^α and on $E^{\alpha-1}$ and such that $|(P - P_0)(u - v)|_\alpha \leqslant q|u - v|_\alpha$ on $\mathcal{M}$, with $q = 1/(2C)$. For $u, v \in \mathcal{M}$ we have that

$$|P_0(u - v)|_\alpha \geqslant |P(u - v)|_\alpha - |(P - P_0)(u - v)|_\alpha \geqslant q|u - v|_\alpha.$$

But then Theorem 1.6 with $\mathcal{K} = \mathcal{A}$ (the implication (Gr)$\to$(FD)) ensures finite dimensionality of the limiting dynamics for equation (1).

Conversely, suppose that the limiting dynamics of (1) is finite-dimensional and the operator A satisfies Hypothesis 1.4. According to Theorem 1.6, the condition (GrF) holds for $\mathcal{K} = \mathcal{A}$, that is, for some $n \geqslant 1$ the spectral projection P_n is bi-Lipschitz on $\mathcal{A}$ in the metric of E^α. Furthermore, $V = P_n\mathcal{A}$ is a compact set in the finite-dimensional subspace P_nE^α. Let $f_0(x) = P_n^{-1}x - x$ for $x \in V$. Then $f_0 \in \mathrm{Lip}(V, Q_nE^\alpha)$. With the help of Lemma 2.4 we extend f_0 to a function $f \in \mathrm{Lip}(P_nE^\alpha, Q_nE^\alpha)$. The graph $\mathcal{M} = \{u \in E^\alpha : u = x + f(x),\ x \in P_nE^\alpha\}$ is a finite-dimensional Lipschitz manifold in E^α, and $\mathcal{M} \supset \mathcal{A}$. The proof of Theorem 1.5 is complete.

§ 3. Proof of Theorem 1.6

We begin with two auxiliary statements, of which the first is a generalization of a similar fact for ODEs. Recall that if an invariant set $\mathcal{N}$ is contained in E^α, then $\mathcal{N} \subset E^1$ and $G(\mathcal{N}) \subset E^\theta$ for any $\theta < 1$.

Lemma 3.1. *Let $\mathcal{N}$ be an invariant subset of E^α and suppose that $|G(u)-G(v)|_\theta \leqslant C|u-v|_\theta$ on $\mathcal{N}$ for some $\theta < 1$, where $C = C(\mathcal{N}, \theta)$. Then the semiflow $\{\Phi_t\}$ is injective on $\mathcal{N}$ and extends to a flow that is Lipschitz in the E^θ-metric.*

Proof. It follows from the invariance of $\mathcal{N}$ that for any $u_0 \in \mathcal{N}$ there is at least one integral curve $u(t)$ with $u(0) = u_0$ that is defined for all $t \in \mathbb{R}$ and lies in $\mathcal{N}$. Integrating equation (1) we find that

$$u(t) = u(0) + \int_0^t G\bigl(u(s)\bigr)\, ds.$$

If $u_0, v_0 \in \mathcal{N}$ and $t > 0$, then

$$|u(t) - v(t)|_\theta \leqslant |u_0 - v_0|_\theta + C \int_0^t |u(s) - v(s)|_\theta\, ds.$$

Gronwall's inequality now gives us the estimate $|u(t)-v(t)|_\theta \leqslant |u_0-v_0|_\theta \exp(C|t|)$. For $t < 0$ the same estimate can be obtained by the substitution $t = -\tau$, and this yields what is required.

The lemma remains valid also for $\theta \geqslant 1$ if it is known a priori that $\mathcal{N} \subset E^\theta$ and $G(\mathcal{N}) \subset E^\theta$.

The following statement about the regularity of the vector field of equation (1) is of independent interest.

Lemma 3.2. *The function $u \to G(\Phi_t u)$ is Lipschitz in the E^α-metric on compact sets $\mathcal{K} \subset E^\alpha$ for any fixed $t > 0$.*

Proof. Let $\Psi(t,u) = \Phi_t u$ for $t > 0$ and $u \in E^\alpha$. Since $F \in C^2(E^\alpha, E)$ according to the hypothesis (H2), it follows from [9] (Corollary 3.4.6) that $\Psi \in C^2(\mathbb{R}^+ \times E^\alpha, E^\alpha)$. For fixed t and u the partial derivative $\Psi_t(t,u)$ is a linear operator acting from $\mathbb{R}$ to E^α according to the rule $\tau \mapsto \tau G(\Phi_t u)$. Since $\Psi \in C^2$, the function $(t,u) \mapsto G(\Phi_t u)$ is a C^1-function with respect to (t,u), and hence for fixed t the function $u \mapsto G(\Phi_t u)$ is a C^1-function and, a fortiori, a locally Lipschitz function from E^α to E^α. But $\mathcal{K}$ is a compact set in E^α and thus $G(\Phi_t)$ is a Lipschitz function on $\mathcal{K}$ for any $t > 0$. The lemma is proved.

Let us now proceed to the proof of Theorem 1.6. We break it up into separate parts.

(VF)$\to$(Fl). It suffices to use Lemma 3.1 with $\mathcal{N} = \mathcal{K}$ and $\theta = \alpha$.

(Fl)$\to$(GrF). Every solution $u(t)$ of equation (1) satisfies the relation

$$u(t) = e^{-A(t-t_0)} u(t_0) + \int_{t_0}^t e^{-A(t-s)} F(u(s))\, ds \tag{9}$$

for $t > t_0$ and $u(t_0) \in E^\alpha$. If $u(0) = u_0 \in \mathcal{K}$, then by the condition (Fl) there is a unique solution $u(t) = \Phi_t u_0 \in \mathcal{K}$ that is defined for all $t \in \mathbb{R}$. We apply to both sides of (9) the projection operator Q_n (with arbitrary $n \geqslant 1$ for the time being) and we set $t = 0$ and $u(0) = u$. Since $Q_n e^{At_0} = e^{At_0} Q_n$ and the compact set $\mathcal{K}$ is bounded in E, the estimate (6) gives us that $|Q_n e^{At_0} u(t_0)|_\alpha = o(1)$ as $t_0 \to -\infty$. This leads to the equality

$$Q_n u = \int_{-\infty}^{0} e^{As} Q_n F(\Phi_s u)\, ds,$$

with the integral convergent in the norm of E^α. By Hypothesis 1.4, there are numbers $\omega_n \in (0, \mu_{n+1})$ such that $\omega_n \to \infty$ as $n \to \infty$ and the constant $M_n = M(n, \alpha, \omega_n)$ in (6) satisfies (7). Using the estimate (6) with $t = -s$ together with (4), we find that

$$|Q_n(u - v)|_\alpha \leqslant L M_n \|Q_n\| \int_{-\infty}^{0} (-s)^{-\alpha} e^{\omega_n s} |\Phi_s u - \Phi_s v|_\alpha\, ds \tag{10}$$

for $u, v \in \mathcal{K} \subset \mathcal{A}$. For the Lipschitz (by the condition (Fl)) flow $\{\Phi_t\}$ on $\mathcal{K}$ we have an exponential (in the metric of E^α) estimate (5) for divergence of trajectories, with some constants $K \geqslant 1$ and $\omega \geqslant 0$. If $\omega_n > \omega$, then (10) implies that $|Q_n(u-v)|_\alpha \leqslant q_n |u - v|_\alpha$, where $q_n = KLM_n \|Q_n\| b_n$ and

$$b_n = \int_{-\infty}^{0} (-s)^{-\alpha} e^{(\omega_n - \omega)s}\, ds,$$

so that $b_n = O(\omega_n^{\alpha-1})$ as $n \to \infty$. According to (7), $\inf_{n \geqslant 1} q_n = 0$. Let n be such that $q_n < 1$. Since $P_n + Q_n = I$, it follows that

$$|Q_n(u - v)|_\alpha \leqslant \frac{q_n}{1 - q_n} |P_n(u - v)|_\alpha, \qquad |u - v|_\alpha \leqslant \frac{1}{1 - q_n} |P_n(u - v)|_\alpha$$

for $u, v \in \mathcal{K}$, and the derivation of (GrF) from (Fl) is complete.

(GrF)$\to$(Gr). In the case when E is reflexive the implication is trivial. But if E is not reflexive, then it suffices to establish that the spectral projection P_n is bounded on $E^{\alpha-1}$. From the integral representations of the linear operators P_n and $A^{\alpha-1}$ in terms of the resolvent $R(\lambda; A)$ (see [11], §§ 3.2.2, 3.3.3) it follows (with the help of the Hilbert identity) that $P_n A^{\alpha-1} = A^{\alpha-1} P_n$ on E. This yields what is required.

(Gr)$\to$(EM). We begin with the existence in $\Pi(\alpha, k)$ (for some $k \geqslant 1$) of a projection P that is bi-Lipschitz on $\mathcal{K}$ in the metric of E^α. It is clear that the set of such projections is open in $\Pi(\alpha, k)$. If E is reflexive, then by Lemma 2.2 (with $\theta = \alpha$ and $\nu = \alpha - 1$) we can assume that P is bounded also on $E^{\alpha-1}$ (in the non-reflexive case this is postulated). Since $PE^\alpha \subset E^{\alpha-1}$ and since all norms are equivalent in the finite-dimensional space PE^α, it follows that

$$|u - v|_\alpha \leqslant C|P(u - v)|_\alpha \leqslant C_1 |P(u - v)|_{\alpha-1} \leqslant C_2 |u - v|_{\alpha-1}$$

for $u, v \in \mathcal{K}$, with a constant C_2 independent of u, v. The opposite inequality (with a different constant) follows from (3), and therefore the metrics of E^α and $E^{\alpha-1}$ are equivalent on $\mathcal{K}$.

(EM)$\to$(VF). Since $|u-v|_\alpha \leqslant C|u-v|_{\alpha-1}$ on $\mathcal{K}$, use of the estimates (3) and (4) gives us that

$$\begin{aligned} |G(u)-G(v)|_{\alpha-1} &\leqslant |A(u-v)|_{\alpha-1} + |F(u)-F(v)|_{\alpha-1} \\ &\leqslant |u-v|_\alpha + c(\alpha-1,0)|F(u)-F(v)| \\ &\leqslant q|u-v|_\alpha \leqslant qC|u-v|_{\alpha-1} \end{aligned} \tag{11}$$

for $u, v \in \mathcal{K}$, with $q = 1 + Lc(\alpha-1,0)$. Thus, the vector field G is Lipschitz in the $E^{\alpha-1}$-metric on the compact invariant set $\mathcal{K} \subset E^\alpha$, and hence by Lemma 3.1 with $\theta = \alpha - 1$ the semiflow $\{\Phi_t\}$ is injective on $\mathcal{K}$ and extends to a flow that is Lipschitz in the $E^{\alpha-1}$-metric. Since the metrics of E^α and $E^{\alpha-1}$ are equivalent on $\mathcal{K}$, this implies the property (Fl), that is, $|\Phi_t u - \Phi_t v|_\alpha \leqslant q(t)|u-v|_\alpha$ on $\mathcal{K}$ for $t \in \mathbb{R}$. By Lemma 3.2 with $t = 1$,

$$|G(u)-G(v)|_\alpha \leqslant C_1|\Phi_{-1}u - \Phi_{-1}v|_\alpha \leqslant C_1 q(-1)|u-v|_\alpha$$

on $\mathcal{K} \subset \mathcal{A}$, and the implication (EM)$\to$(VF) is proved.

Accordingly, we have shown that for an arbitrary compact invariant set $\mathcal{K}$ in E^α the conditions (VF), (Fl), (GrF), (Gr), and (EM) are equivalent. It remains to establish their connection with the property (FD) of finite dimensionality of the dynamics.

(FD)$\to$(Fl). This is a direct consequence of Definition 1.1 (see Remark 1.2).

(Gr)$\to$(FD). As already shown, (Gr) implies (VF). According to (Gr) there is a projection P of finite rank N that is bounded on E^α and that implements a Lipschitz embedding $\mathcal{K} \to PE^\alpha$. Further, the equation (1) $\partial_t u = G(u)$ on $\mathcal{K}$ is projected into the equation $\dot{x} = h_0(x)$ for $x = Pu$, with Lipschitz (by the condition (VF)) function $h_0(x) = PG(P^{-1}x)$ on the compact set $V = P\mathcal{K}$ in PE^α. Identifying PE^α with $\mathbb{R}^N$ and using Lemma 2.4 to extend the function $h_0 \in \operatorname{Lip}(V, \mathbb{R}^N)$ to a function $h \in \operatorname{Lip}(\mathbb{R}^N, \mathbb{R}^N)$, we get an ODE $\dot{x} = h(x)$ in $\mathbb{R}^N$ with phase flow $\{S_t\}$. Since $\partial_t(P\Phi_t P^{-1}x)\big|_{t=0} = h_0(x)$ for $x \in V$, it follows (by the uniqueness theorem for ODEs) that $S_t x = P\Phi_t P^{-1}x$ on V. But $x = Pu$, and therefore $S_t Pu = P\Phi_t u$ for $u \in \mathcal{K}$, which implies the finite dimensionality of the dynamics on $\mathcal{K}$. The proof of Theorem 1.6 is complete.

§ 4. Regularity of the vector field on the attractor

In connection with the necessary and sufficient conditions obtained in Theorem 1.6 for finite dimensionality of the limiting dynamics, it is of interest to determine how close these conditions are to real characteristics of arbitrary equations of the form (1). This can be done, at least in the cases (EM) and (VF). It turns out that the operators A and $G = -A + F$ are always Hölder on the attractor $\mathcal{A}$ in the metrics of $E^{\alpha-1}$ and E^α, respectively, with the exponent depending only on α.

The question of the regularity on the attractor of the vector field of a semilinear parabolic equation has already been discussed in the context of the theory of exponential attractors in [13], and also in Robinson's paper [14]. It is known, in particular, that the operator G is Hölder on $\mathcal{A}$ in the metric of E with exponent $\beta = 1 - \theta^{-1}$ if the attractor $\mathcal{A}$ is bounded in E^θ with $\theta > 1$. However, here we are considering the question of regularity of the vector field G on $\mathcal{A}$ in the metric of E^α.

As above, we assume the basic hypotheses (H1)–(H3).

Theorem 4.1. *The following estimates hold for* $u, v \in \mathcal{A}$:

$$|A(u-v)|_{\alpha-1} \leqslant C|u-v|_{\alpha-1}^\beta, \tag{12a}$$

$$|G(u)-G(v)|_\alpha \leqslant C|u-v|_\alpha^\beta, \tag{12b}$$

where $\beta = (1-\alpha)/(2-\alpha)$ *and* $C = C(\mathcal{A})$.

We note that $\beta = 1/2$ for $\alpha = 0$, $\beta = 1/3$ for $\alpha = 1/2$, and $\beta \to 0$ as $\alpha \to 1$. The estimates (12) are actually valid for a broad class of subsets of the space E^α.

Lemma 4.2. *Let* $\beta = (1-\alpha)/(2-\alpha)$. *If* $\mathcal{N}$ *is a bounded subset of* E^1, *then the estimate* (12a) *holds on* $\mathcal{N}$. *If* $\mathcal{N}$ *is a compact subset of* E^α *and the set* $G(\mathcal{N})$ *is bounded in* E^1, *then the estimate* (12b) *holds on* $\mathcal{N}$.

Proof. As in [13] and [14], we use interpolation in the scale $\{E^\theta\}$ of Banach spaces (see [9], § 1.4). In our case this yields

$$|u-v|_\alpha \leqslant K|u-v|_{\alpha-1}^\beta |u-v|_1^{1-\beta}$$

for $u, v \in \mathcal{N}$, with a constant $K = K(A)$. If $\mathcal{N}$ is bounded in E^1, then this yields the estimate (12a). Similarly,

$$|G(u)-G(v)|_\alpha \leqslant K|G(u)-G(v)|_{\alpha-1}^\beta |G(u)-G(v)|_1^{1-\beta}$$

if $G(u), G(v) \in E^1$. But $|G(u)-G(v)|_{\alpha-1} \leqslant q|u-v|_\alpha$ by (11), and this yields the estimate (12b) on $\mathcal{N}$ when the set $G(\mathcal{N})$ is bounded in E^1. The lemma is proved.

Thus, to prove Theorem 4.1 we need boundedness of the sets $\mathcal{A}$ and $G(\mathcal{A})$ in E^1.

Lemma 4.3. *The set* $G(\mathcal{A})$ *is bounded in* E^1.

It follows immediately from Theorem 3.5.2 in [9] that $G(\mathcal{A})$ is bounded in E^θ for any $\theta < 1$. If E is a Hilbert space, A is self-adjoint, and $\alpha \leqslant 1/2$, then the boundedness of $G(\mathcal{A})$ in E^1 is a consequence of a more general assertion obtained by Chueshov ([5], Lemma 12.1).

Proof of Lemma 4.3. Let $\Psi(t, v) = \Phi_t v$ for $t > 0$ and $v \in E^\alpha$. The function F is in $C^2(E^\alpha, E)$, and hence ([9], Corollary 3.4.6) $\Psi \in C^2(\mathbb{R}^+ \times E^\alpha, E^\alpha)$. For given $t > 0$ and $v \in E^\alpha$ the partial derivative $\Psi_{tt}(t, v)$ is a bilinear operator acting from $\mathbb{R} \times \mathbb{R}$ to E^α according to the rule $(\tau_1, \tau_2) \to \tau_1\tau_2\partial_{tt}u(t)$, where $u(t) = \Phi_t v$, $\partial_t u = G(\Phi_t v)$, $\partial_{tt}u = (-A + F'(\Phi_t v))G(\Phi_t v) \equiv f(t, v)$, and F' is the Fréchet derivative. Since $\Psi \in C^2$, it follows that the map f is continuous from $\mathbb{R}^+ \times E^\alpha$ to E^α. Thus, for fixed t (for example, for $t = 1$) the function $f(t, \cdot)$ is continuous from E^α to E^α. But $\mathcal{A}$ is a compact set in E^α, and therefore this function is bounded on $\mathcal{A}$ in the metric of E^α and, a fortiori, in the metric of E.

Further, $f(t,v) = (-A + F'(u))G(u)$, where $u = \Phi_t v$. Lemma 3.2 gives us that the set $G(\mathcal{A})$ is bounded in E^α, while the condition $F \in C^2(E^\alpha, E)$ implies that the norms of the linear operators $F'(u)$ from E^α to E are uniformly bounded with respect to $u \in \mathcal{A}$. Thus, $|F'(u)G(u)| \leqslant M$ for $u \in \mathcal{A}$ and the boundedness of the norm $|f(t,v)|$ on $\mathcal{A}$ implies an estimate $|AG(u)| = |G(u)|_1 \leqslant K$ for $u \in \mathcal{A}$. The lemma is proved.

Theorem 4.1 is easy to derive from Lemmas 4.2 and 4.3. Indeed, $|G(u)|_1 \leqslant K$ by Lemma 4.3, and, a fortiori, $|G(u)| \leqslant K_1$ for $u \in \mathcal{A}$. The compact set $\mathcal{A}$ is bounded in E^α, and thus $|F(u)| \leqslant K_2$ on $\mathcal{A}$ by (4). Since $A = F - G$, this implies that the attractor $\mathcal{A}$ is bounded in E^1 and it remains to use Lemma 4.2. The proof of Theorem 4.1 is complete.

Suppose next that the basic space E is a Hilbert space. We shall determine when the estimates (12) are true with arbitrary exponent $0 < \beta < 1$ by using the classical results of Ladyzhenskaya [15] on well-posedness in the class of bounded solutions of the inverse Cauchy problem for semilinear parabolic equations with self-adjoint linear part. We recall that in its time the article cited (along with [16]) laid the foundation for a realization of Hopf's idea (mentioned above) about the finite-dimensional character of the limiting regimes of such equations. It was in [15] that the compact attractor $\mathcal{A}$ was proposed as the fundamental object of the theory of infinite-dimensional evolution systems with dissipation, and a kind of finite dimensionality (different from that discussed here) of the dynamics on $\mathcal{A}$ was established as a general property.

Moreover, the condition (2.8) in [15] enables us to single out a class of equations (1) with continuous phase flow $\{\Phi_t\}$ on the attractor that admits (as a direct consequence of the inequality (2.11) in the same article) the estimate

$$|\Phi_{-2t}u - \Phi_{-2t}v| \leqslant M(t,\sigma)|u - v|^{\sigma\gamma} \tag{13}$$

for $u, v \in \mathcal{A}$ and $t > 0$, with an arbitrary $\sigma \in (0,1)$ and with $\gamma = e^{-2\varkappa t}$, $\varkappa = \varkappa(\mathcal{A})$. Using the relations

$$|G(u) - G(v)|_\alpha \leqslant K(t)|\Phi_{-t}u - \Phi_{-t}v|_\alpha \quad \text{(Lemma 3.2)},$$
$$|\Phi_{-t}u - \Phi_{-t}v|_\alpha \leqslant K_1(t)|\Phi_{-2t}u - \Phi_{-2t}v| \quad \text{([17], Lemma 5.2)},$$

we get from (13) and (3) the estimate (12b) for the vector field G on $\mathcal{A}$ for each $\beta < 1$, with $C = C(\beta, \mathcal{A}) \to \infty$ as $\beta \to 1$. An analogous argument works for (12a) with $\beta < 1$.

We now consider the Navier–Stokes system in a rectangle $\Omega \subset \mathbb{R}^2$ with a periodicity condition on $\partial\Omega$. We assume sufficient (but finite) smoothness of the external force $f(x)$ with respect to $x \in \Omega$. It is known that such a system can be written in the form (1) with $A^* = A$ and $F \in C^\infty(E^\alpha, E)$ with $\alpha > 1/2$ and that it has a compact attractor $\mathcal{A} \subset E^\alpha$ if E is taken to be the subspace of divergence-free vector-valued functions with zero mean in $L^2(\Omega; \mathbb{R}^2)$. In this situation it is possible first to use techniques in [15] to establish the relations (2.8) and (2.11) of that article and then to derive the estimates (13) and (12) with $\beta < 1$. The existence of an inertial manifold for this problem has not yet been proved. At the same time an 'almost Lipschitz' estimate of the vector field on the attractor allows us to expect that some additional requirements (of analyticity type for the external force f) can ensure the finite dimensionality of the limiting dynamics of the two-dimensional Navier–Stokes equations on a torus in the sense of Definitions 1.3 and 1.1.

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All-Russian Institute of Scientific and Technological Information,
Russian Academy of Sciences, Moscow

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