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Finite-dimensional dynamics on attractors of non-linear parabolic equations

A. V. Romanov

Abstract. We show that one-dimensional semilinear second-order parabolic equations have finite-dimensional dynamics on attractors. In particular, this is true for reaction-diffusion equations with convection on $(0, 1)$.

We obtain new topological criteria for a class of dissipative equations of parabolic type in Banach spaces to have finite-dimensional dynamics on invariant compact sets. The dynamics of these equations on an attractor $\mathcal{A}$ is finite-dimensional (can be described by an ordinary differential equation) if $\mathcal{A}$ can be embedded in a finite-dimensional C^1-submanifold of the phase space.

Introduction

This paper, like our earlier paper [1], deals with a new approach to the study of the limiting behaviour of semilinear parabolic equations,

$$\partial_t u = -Au + F(u), \qquad u = u(t), \tag{1}$$

in a Banach space X with an unbounded linear sectorial operator A and "relatively weak" non-linearity F. We assume that (1) generates a smooth dissipative semiflow in X^α, where $0 \leqslant \alpha < 1$ and $\{X^\theta\}_{\theta\in\mathbb{R}}$ is the scale of Banach spaces determined by A [2].

We discuss conditions under which the dynamics of (1) on an invariant compact set $\mathcal{K} \subset X^\alpha$ can be described by an ordinary differential equation in $\mathbb{R}^n$ with a Lipschitzian vector field, which implies that the phase semiflow on $\mathcal{K}$ is Lipschitz-conjugate to the corresponding phase semiflow on some compact set $\mathcal{K}_1 \subset \mathbb{R}^n$ invariant under the ordinary differential equation. In this case we say that we have *finite-dimensional dynamics on* $\mathcal{K}$. If equation (1) has a compact attractor $\mathcal{A} \subset X^\alpha$ (that is, an invariant set that attracts the balls $\mathcal{B} \subset X^\alpha$ uniformly as $t \to +\infty$), then our main attention will be paid to finite-dimensional limiting dynamics of (1), by which we mean (see [1]) finite-dimensional dynamics of (1) on $\mathcal{A}$.

The conjecture that the final behaviour of non-linear parabolic equations is finite dimensional was made by E. Hopf [3]. Foias and Prodi [4] and Ladyzhenskaya [5] were the first to obtain concrete results in this area (stating that "there are finitely

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many determining modes"). The existence of compact attractors $\mathcal{A}$ was established in [5] for a wide class of problems (1). At that time the finite-dimensionality of the dynamics on $\mathcal{A}$ was treated by Ladyzhenskaya as a means of reconstructing the full trajectories $\{u(t)\}_{t\in\mathbb{R}} \subset \mathcal{A}$ from their projections on a suitable finite-dimensional subspace $Y \subset X^\alpha$. Convenient tools for proving that invariant compact sets are finite-dimensional (1) were provided by Mallet-Paret's general theorem [6] on smooth completely continuous maps in Hilbert space, as well as by its Banach version [7]. Various methods for estimating the (Hausdorff, fractal or Lyapunov) dimension of attractors of evolution equations can be found in [8]–[10]. In the recent paper of Chueshov [11] the problem of finite-parametric tracing of trajectories of distributed dynamical systems is treated from a unified point of view.

Mane [12] considered equation (1) with a self-adjoint operator A in Hilbert space and suggested for it (see also [10], [13]) the so-called *spectral jump condition*, which enables one to construct an inertial manifold $\mathcal{M} \subset X^\alpha$, that is, a smooth or Lipschitzian finite-dimensional invariant surface that contains an attractor $\mathcal{A}$ and attracts the balls $\mathcal{B} \subset X^\alpha$ at an exponential rate. Here $\mathcal{M}$ is a graph, and the restriction of equation (1) to $\mathcal{M}$ gives an inertial form, that is, an ordinary differential equation in $\mathbb{R}^n$ that models both the exact behaviour of solutions $u(t)$ on $\mathcal{A}$ and the asymptotics of the phase dynamics of (1) in X^α as $t \to +\infty$. The existence of an inertial manifold ("asymptotically finite-dimensional dynamics") is a somewhat stronger property than finiteness of dimension of the limiting dynamics. Unfortunately, the spectral jump condition, which implies considerable sparseness of the spectrum of A, turned out to be very restrictive, and the theory of inertial manifolds has, in fact, reached a state of deadlock.

On the other hand, there are reasons to believe that it is sometimes easier to establish the finite-dimensionality of the dynamics on the attractor of a concrete dissipative system than to prove that it has an inertial manifold. The first example (but certainly not the last) supporting this view is provided by the one-dimensional parabolic equation

$$u_t = u_{xx} + f(x, u, u_x), \qquad x \in (0, 1), \tag{2}$$

with separated or periodic boundary conditions and a smooth function f such that the mixed problem has a global solution (for $t > 0$) satisfying suitable estimates. The lower semibounded self-adjoint operator $A = -\partial_{xx}$ in $X = L^2(0, 1)$ determines a Hilbert scale of spaces $\{X^\theta\}$. We assume that there is an $\alpha \in (3/4, 1)$ such that equation (2) generates a smooth semiflow in X^α and has a compact attractor $\mathcal{A} \subset X^\alpha$. One of the main results of this paper (Theorem 3.3) states that the phase dynamics of (2) on $\mathcal{A}$ is finite-dimensional. However, it is as yet unknown whether this semiflow always has an inertial manifold. According to [1], the finite-dimensionality of the limiting dynamics of [2] implies that the vector field of the equation is Lipschitzian on $\mathcal{A}$ in the X^α-metric, the semiflow can be extended to a flow on $\mathcal{A}$ that is Lipschitzian in the X^α-metric, and the attractor $\mathcal{A}$ is a part of a finite-dimensional Lipschitzian manifold (of graph type) $\mathcal{M} \subset X^\alpha$. On the other hand, equation (2) on a circle has solutions $u(t, \cdot\,)$ periodic in t, and the periods of these solutions are bounded below by a positive constant $c = c(f)$, which follows from the finite-dimensionality of the dynamics on the attractor.

Systems of equations (2) with $u(0) = u(1) = 0$ and $(d(x)u_x)_x$ instead of u_{xx} also have finite-dimensional limiting dynamics if the smooth non-homogeneous "diffusion coefficient" $d(x) > 0$, defined on $[0,1]$, is the same for all components of the system.

The proof of Theorem 3.3 is based on the results of §§ 1, 2 for an abstract equation (1), which are of some independent interest. Let A be a discrete sectorial operator on X, assume that the function F belongs to $C^2(X^\alpha, X)$ with some $\alpha \in [0,1)$ and is bounded on the balls in X^α, and let $|\cdot|_\alpha$ be the norm in X^α. Theorem 1.4 establishes two criteria (new in comparison with [1]) for the dynamics of (1) to be finite-dimensional on the invariant compact sets $\mathcal{K} \subset X^\alpha$. The assumption in the case of the first criterion is the relative compactness of the set $w = (u-v)/|u-v|_\alpha$, $u, v \in \mathcal{K}$, $u \neq v$ in X^α. In the second it is assumed that for any $w \in \mathcal{K}$ one can find an X^α-neighbourhood $\mathcal{V} \supset w$ and a finite-dimensional projector P continuous in X^α and such that

$$|u-v|_\alpha \leqslant c|P(u-v)|_\alpha$$

on $\mathcal{V} \cap \mathcal{K}$, $c = c(\mathcal{K}, w, P)$. If the space X is not reflexive, then we assume that P is continuous in $X^{\alpha-1}$. Theorem 1.5 describes the relation between the finite-dimensionality of the dynamics on the invariant compact set $\mathcal{K} \subset X^\alpha$ and the (identical) embeddability of $\mathcal{K}$ in a sufficiently regular finite-dimensional submanifold $\mathcal{M} \subset X^\alpha$. Namely, if $\mathcal{M}$ is C^1-smooth, then the phase dynamics is finite-dimensional on $\mathcal{K}$. The converse is true if $\mathcal{M}$ is a Lipschitzian manifold. Let us emphasize that the manifold $\mathcal{M}$ is not assumed to be invariant.

Note that some of the above statements hold under the additional assumption that the finite-dimensional invariant subspaces of A, ordered in suitable way, have the basis property in X^α.

All these constructions are of a topological nature, but in the Hilbert case Theorems 2.3 and 2.8 provide analytical conditions on the vector field $G(u) = F(u) - Au$ of equation (1) under which its limiting dynamics is finite-dimensional. These conditions involve the decomposition

$$G(u) - G(v) = \bigl(B_0(u,v) - B(u,v)\bigr)(u-v)$$

on the attractor $\mathcal{A} \subset X^\alpha$, where $B_0(u,v)$ is the field of continuous linear maps $X^\alpha \to X^\alpha$ and $B(u,v)$ is a field of unbounded sectorial linear operators on X similar to normal ones. We also assume that the set $\Sigma = \bigcup_{u,v\in\mathcal{A}} \sigma(B(u,v))$ is sufficiently sparse, but this condition is less restrictive than the condition on $\sigma(A)$ in the spectral jump condition if $\alpha \neq 0$. Under some technical assumptions on B_0 and B we establish that the second criterion in Theorem 1.4 is applicable, which implies that the dynamics on $\mathcal{A}$ is finite-dimensional.

This scenario can be successfully realized for one-dimensional semilinear parabolic equations (2), but the corresponding passage to dimensions $\geqslant 2$ remains problematical even in the "simple" situation when $f = f(x,u)$. We have not succeeded as yet in establishing the finite-dimensionality of the limiting dynamics for the two-dimensional Navier–Stokes system, although the arguments in [1], § 4, lead us to hope that this might be proved in the case of periodic boundary conditions. Further efforts are required to weaken the assumptions of Theorems 2.3 and 2.8

in order to enlarge the list of equations of parabolic type with finite-dimensional dynamics on the attractors.

The natural question of whether equations (1) can have solutions whose limiting dynamics is not finite-dimensional seems to be rather difficult. Until recently we have not been able to answer the corresponding question concerning the asymptotic finite-dimensionality of the phase dynamics. It was only in [14] that we succeeded in producing an example of equation (1) with no inertial C^1-manifold $\mathcal{M} \subset X^\alpha$. We have not succeeded as yet in constructing a counterexample of this kind in the case when the class of smoothness of $\mathcal{M}$ is replaced by Lip.

It is important to cite very interesting papers by Kamaev [15], [16], in which an invariant C^1-continuous family of smooth stable manifolds of finite codimension is constructed for the attractors of equations similar to (2) and the corresponding systems of equations. It would be interesting to elicit (in the general case) the relation between the existence of such a family of manifolds and the finite-dimensionality of the limiting dynamics of the evolution problems (1).

We do not produce here any quantitative estimates for the phase dimension of the ordinary differential equation describing the limiting dynamics of (1), nor do we compare them with the well-known estimates for the dimensions of attractors and inertial manifolds (if the latter exist). This promising topic exceeds the limits of this paper and may serve an object of further investigation.

§ 1. Topological conditions

Let us specify some concepts concerning (1) and recall some well-known properties of these equations (see, in particular, [2]).

Let X be a separable infinite-dimensional Banach space with norm $|\cdot|$. Let $\sigma(\,\cdot\,)$, $\|\cdot\|$ and $R(\lambda;\,\cdot\,)$ be the spectrum, norm, and resolvent of linear operators on X. The closed linear operator A in (1) with a dense domain $\mathcal{D}(A) \subset X$ is assumed to be sectorial and discrete. The former assumption means that there are $k > 0$ and $\lambda_0 \in \mathbb{R}$ such that $\sigma(A)$ is contained in the sector $|\operatorname{Im}\lambda| < k\operatorname{Re}(\lambda - \lambda_0)$ of the complex plane $\mathbb{C}$ and

$$\|R(\lambda; A)\| \leqslant M/(1 + |\lambda - \lambda_0|), \qquad M = M(A, k, \lambda_0)$$

outside this sector. The latter assumption (of discreteness) means that $R(\lambda; A)$ is compact. In what follows we assume that $\lambda_0 = 0$ or, equivalently, $\operatorname{Re}\sigma(A) > 0$ (that is, $\operatorname{Re}\lambda > 0$ on $\sigma(A)$). Hence, the powers A^θ and the Banach spaces $X^\theta = \mathcal{D}(A^\theta)$ with the norm $|u|_\theta = |A^\theta u|$ are well defined for all $\theta \in \mathbb{R}$. For $\theta < 0$ the operators A^θ are completely continuous in X and $\mathcal{D}(A^\theta)$ is the completion of X in the norm $|\cdot|_\theta$. We have $X^0 = X$ and $X^1 = \mathcal{D}(A)$. For $\beta < \theta$ the embeddings $X^\theta \subset X^\beta$ are absolutely continuous. For $\beta, \theta \in \mathbb{R}$ the operators A^β map $X^{\theta+\beta}$ isometrically onto X^θ.

We denote by $BC^\nu(Y_1, Y_2)$, $\nu \in \mathbb{Z}^+$ ($BC(Y_1, Y_2)$) the class of C^ν-smooth (continuous) maps $\Pi\colon Y_1 \to Y_2$ bounded on balls, where Y_1 and Y_2 are Banach spaces.

If $\alpha \in [0, 1)$ and $F \in BC^2(X^\alpha, X)$, then the Cauchy problem for (1) with $u(0) = u_0 \in X^\alpha$ has a strong local solution $u(t) \in C^2((0, t^*), X^\alpha) \cap C\bigl([0, t^*), X^\alpha\bigr)$ for $t^* = t^*(u_0) > 0$. In fact, $u(t) \in X^1$ on $(0, t^*)$. We also assume that equation (1) is dissipative in X^α, that is, it has global solutions $u(t) = \Phi_t u_0$ for $t > 0$ and there is an (absorbing) ball $\mathcal{B}_0 \subset X^\alpha$ such that $\Phi_t\mathcal{B} \subset \mathcal{B}_0$ for every ball $\mathcal{B} \subset X^\alpha$ if $t > t_0(\mathcal{B})$. Then the phase semiflow $\{\Phi_t\}_{t\geqslant 0}$ is a map $(0,\infty)\times X^\alpha \to X^\alpha$ of class C^2. If $\mathcal{U} \subset X^\alpha$, then $\Phi_t\mathcal{U}$ is bounded in X^θ for $t > 0$, $\alpha < \theta < 1$ (which implies that $\Phi_t\mathcal{U}$ is relatively compact in X^α). This can be deduced from the fact that the current tube $\bigcup_{0\leqslant\tau\leqslant t}\Phi_\tau\mathcal{U}$ is X^α-bounded by means of the arguments used in [2], Theorem 3.3.6. Hence, the evolution operator Φ_t is compact on every ball $\mathcal{B} \subset X^\alpha$ if $t > t_0(\mathcal{B})$.

So we assume that the following three hypotheses hold:

(H1) the linear operator A is discrete and sectorial, the spectrum $\sigma(A)$ is countable and $\operatorname{Re}\sigma(A) > 0$,

(H2) $F \in BC^2(X^\alpha, X)$ for some $\alpha \in [0,1)$,

(H3) equation (1) is dissipative in X^α.

The assumption that $\sigma(A)$ is countable is purely technical. In Hilbert space, (H1) holds for any discrete normal operator whose spectrum is contained in the sector $|\operatorname{Im}\lambda| < k\operatorname{Re}\lambda$ with $k = \mathrm{const} > 0$ (for example, for any discrete positive-definite operator).

We denote by $G(u)$ the vector field $F(u) - Au$ of equation (1). The set $\mathcal{U} \subset X^\alpha$ is invariant if $\Phi_t\mathcal{U} = \mathcal{U}$ for $t > 0$ (in fact, we have $\mathcal{U} \subset X^1$). Bounded invariant subsets of X^α are relatively compact. If (H1)–(H3) hold, then the phase semiflow $\{\Phi_t\}$ has (see [8]–[10]) a compact attractor $\mathcal{A}$, which is the maximal bounded invariant subset of X^α. It was shown in [1], § 4, that the function $G\colon \mathcal{A} \to X^\alpha$ is Hölderian in the X^α-metric and $|u|_1 \leqslant \mathrm{const}$ on $\mathcal{A}$. The last assertions remain valid if $F \in BC^1(X^\alpha, X)$, which can be proved by writing (1) in integral form. Theorem 1.4 in [1] implies that the following lemma holds.

Lemma 1.1. *The function $A\colon \mathcal{A} \to X$ is Hölderian in the X^α-metric.*

Remark 1.2. Hypothesis (H1) holds for $A\colon X^1 \to X$ and $A\colon X^{1+\beta} \to X^\beta$ with $\beta > 0$. Replacing (X^α, X) in (H2), (H3) by $(X^{\alpha+\beta}, X^\beta)$, we can transfer the above properties of the dynamics of (1) to the phase space $X^{\alpha+\beta}$. The same is true for the subsequent constructions.

The existence of an absorbing ball for the semiflow $\{\Phi_t\}$ is needed only to guarantee the existence of a compact attractor.

Remark 1.3. If we restrict ourselves to the study of the dynamics of (1) on arbitrary invariant compact sets $\mathcal{K} \subset X^\alpha$, then (H3) can be replaced by the hypothesis that the solutions $u(t)$ can be extended to $(0,\infty)$ for all initial values $u_0 \in X^\alpha$.

Both of these remarks are also applicable to the results of [1].

For $a > 0$ we denote by $\mathcal{P}_a$ the finite-dimensional spectral projector of the operator A on X corresponding to the part of spectrum with $\operatorname{Re}\lambda < a$. The projectors $\mathcal{P}_a$ commute with A^α and are continuous in X and X^α.

We say (see [1], Definition 1.1) that the phase dynamics of (1) on the invariant compact set $\mathcal{K} \subset X^\alpha$ is *finite-dimensional* if there are ordinary differential equations with a Lipschitzian vector field and a resolving flow $\{\varphi_t\}$ in $\mathbb{R}^n$, and a Lipschitzian embedding $\Psi\colon \mathcal{K} \to \mathbb{R}^n$ such that

$$\Psi\Phi_t u = \varphi_t\Psi u, \qquad u \in \mathcal{K}, \quad t \geqslant 0.$$

If $\mathcal{K} = \mathcal{A}$, then the assertion that equation (1) has finite-dimensional limiting dynamics has the same meaning.

By Theorem 1.6 in [1], the following assertions are equivalent:

(FD) the phase dynamics on $\mathcal{K}$ is finite-dimensional,

(VF) $|G(u) - G(v)|_\alpha \leqslant c|u - v|_\alpha$ for $u, v \in \mathcal{K}$, $c = c(\mathcal{K})$,

(Fl) the semiflow $\{\Phi_t\}$ on $\mathcal{K}$ is injective and can be extended to a flow that is Lipschitzian in the X^α-metric,

(GrF) there is an $a > 0$ such that the estimate $|u - v|_\alpha \leqslant c|\mathcal{P}_a(u - v)|_\alpha$ holds for $u, v \in \mathcal{K}$, $c = c(\mathcal{K}, a)$,

(Gr) there is a finite-dimensional projector P continuous in X^α (and in $X^{\alpha-1}$ if X is non-reflexive) and such that $|u-v|_\alpha \leqslant c|P(u-v)|_\alpha$ for $u, v \in \mathcal{K}$, $c = c(\mathcal{K}, P)$,

(EM) the metrics of X^α and $X^{\alpha-1}$ are equivalent on $\mathcal{K}$.

The theorem cited establishes the logical cycle (VF)$\to$(Fl)$\to$(GrF)$\to$(Gr)$\to$(EM)$\to$(VF) and the implications (FD)$\to$(Fl) and (Gr)$\to$(FD). The implication (Fl)$\to$(GrF) holds under an additional condition [1], Proposition 1.4, on the operator A stated in terms of properties of the semigroup $\{\exp(-tA)\}_{t \geqslant 0}$.

Let us state two more criteria for the finite-dimensionality of the dynamics on invariant compact sets.

Theorem 1.4. *Let*

$$\lim_{a \to \infty} |u - \mathcal{P}_a u|_\alpha = 0 \tag{3}$$

for all $u \in X^\alpha$, and let $\mathcal{K}$ be an invariant compact set of equation (1) *in X^α. Then the dynamics on $\mathcal{K}$ is finite-dimensional if and only if one of the following equivalent assertions holds*:

(KC) *the set $\mathcal{K}^0$ of points $w = (u-v)/|u-v|_\alpha$ with $u, v \in \mathcal{K}$, $u \neq v$, is relatively compact in X^α*,

(GrL) *for any $w \in \mathcal{K}$ one can find X^α-neighbourhoods $\mathcal{V} \supset w$ and a finite-dimensional projector P in X^α* (*and in $X^{\alpha-1}$ if X is non-reflexive*) *such that $|u - v|_\alpha \leqslant c|P(u - v)|_\alpha$ on $\mathcal{V} \cap \mathcal{K}$, $c = c(\mathcal{K}, w, P)$.*

Let us emphasize that the rank of P in (GrL) may depend on w.

Proof. Let us establish the logical chain (FD)$\to$(KC)$\to$(GrL)$\to$(FD). We know that (FD)$\to$(Fl), (GrF)$\to$(Gr)$\to$(FD) and (EM)$\to$(Fl). It is obvious that (Gr)$\to$(GrL). Hence, it is sufficient to prove the implications (Fl)$\to$(KC)$\to$(GrF) and (GrL)$\to$(EM).

(Fl)$\to$(KC). Let $\alpha < \theta < 1$. For $t > 0$ we have the estimate (see [17], Lemma 5.2) $|\Phi_t u - \Phi_t v|_\theta \leqslant c_t|u - v|_\alpha$ on $\mathcal{K}$ with $c_t = c(\mathcal{K}, \theta; t)$. Using property (Fl) of the semiflow $\{\Phi_t\}$, we obtain that

$$|u - v|_\theta \leqslant c_1|\Phi_{-1}u - \Phi_{-1}v|_\alpha \leqslant N|u - v|_\alpha, \qquad N = \text{const}\,.$$

Therefore, $\bigl|(u-v)/|u-v|_\alpha\bigr|_\theta \leqslant N$ on $\mathcal{K}$ and the set $\mathcal{K}^0$ is bounded in X^θ. Hence, this set is relatively compact in X^α.

(KC)$\to$(GrF). Formula (3) implies that $\|I - \mathcal{P}_a\|_\alpha \leqslant \text{const}$.[1] It follows from Ascoli's theorem that $\mathcal{P}_a \to I$ (as $a \to \infty$) uniformly on the relatively compact set

[1]Here and below $I = \text{id}$.

$\mathcal{K}^0 \subset X^\alpha$ and $|h - \mathcal{P}_a h|_\alpha \leqslant \varepsilon_a |h|_\alpha$, where $\varepsilon_a \to 0$ and $h = u - v$ for $u, v \in \mathcal{K}$. If $\varepsilon_a < 1$ and $c = (1-\varepsilon_a)^{-1}$, then $|h - \mathcal{P}_a h|_\alpha \leqslant c\varepsilon_a |\mathcal{P}_a h|_\alpha$ and $|u-v|_\alpha \leqslant c|\mathcal{P}_a(u-v)|_\alpha$ on $\mathcal{K}$.

(GrL)→(EM). Let $\delta > 0$ be the Lebesgue number [18], 2.13.4, of the open covering of the compact set $\mathcal{K}$ by the sets $\mathcal{U}(w) = \mathcal{V} \cap \mathcal{K}$ (we assume that this covering is finite). Then any two points $u, v \in \mathcal{K}$ with $|u - v|_\alpha < \delta$ belong to the same $\mathcal{U}(w)$. The arguments used in the proof of the implication (Gr)→(EM) in [1], Theorem 1.6, yield the estimate $|u-v|_\alpha \leqslant c|u-v|_{\alpha-1}$ on the X^α-closure of each of the sets $\mathcal{U}(w)$ with the same constant $c > 0$. A similar relation holds for $u, v \in \mathcal{K}$ such that $|u - v|_\alpha \geqslant \delta$. (Otherwise we would have an absurd situation: one could find sequences $\{u_l\}, \{v_l\} \subset \mathcal{K}$ converging in the X^α-metric and such that $|u_l - v_l|_\alpha \geqslant \delta$, but $|u_l - v_l|_{\alpha-1} \to 0$ as $l \to \infty$.) Finally, the inequality $|u - v|_\alpha \geqslant c'|u - v|_{\alpha-1}$ holds on $\mathcal{K}$ since the embedding $X^\alpha \subset X^{\alpha-1}$ is continuous, which completes the proof of Theorem 1.4.

Now we have the extended logical cycle (VF)→(Fl)→(KC)→(GrF)→(Gr)→(GrL)→(EM)→(VF) along with the implications (FD)→(Fl) and (Gr)→(FD) (assumption (3) was used only in the proof of the implication (KC)→(GrF)). In fact, assumption (3) means that the finite-dimensional invariant subspaces of the operator A corresponding to the spectral sets $\{\lambda \in \sigma(A)\colon \operatorname{Re}\lambda = \text{const}\}$ arranged in increasing order of $\operatorname{Re}\lambda$ have the basis property (in X^α). This assumption holds if A is a spectral operator on X (see [19]) (in which case it is a spectral operator on X^α as well). If X is a Hilbert space, then this assumption holds if the operator A is similar to a normal operator.

We shall now discuss a condition under which the phase dynamics on the invariant compact set $\mathcal{K} \subset X^\alpha$ is finite-dimensional. This condition is of special interest. Theorem 1.5 in [1] establishes a relation between the embeddability of $\mathcal{K}$ in a (sufficiently regular) finite-dimensional submanifold $\mathcal{M} \subset X^\alpha$ and the finite-dimensionality of the dynamics on $\mathcal{K}$. Our next theorem is a version of that theorem. This version is definitive as far as the order of smoothness of $\mathcal{M}$ is concerned.

Theorem 1.5. *Let $\mathcal{K}$ be an invariant compact set of equation* (1) *in X^α and assume that* (3) *holds for the operator A. If $\mathcal{K} \subset \mathcal{M}$, where $\mathcal{M}$ is a finite-dimensional C^1-submanifold in X^α, then the phase dynamics on $\mathcal{K}$ is finite-dimensional. Conversely, if the dynamics on $\mathcal{K}$ is finite-dimensional, then there is an $a > 0$ such that the set $\mathcal{K}$ belongs to the graph of a uniformly Lipschitzian map from $\mathcal{P}_a X^\alpha$ to $(I - \mathcal{P}_a)X^\alpha$.*

This theorem was announced in [20]. It differs from the cited result of [1] in that the order of class of smoothness of $\mathcal{M}$ in its hypotheses is lowered from C^2 to C^1 and in certain details of the assumptions on the linear operator A. We see that the limiting dynamics of (1) is finite-dimensional if the attractor $\mathcal{A}$ can be embedded in a finite-dimensional C^1-manifold $\mathcal{M} \subset X^\alpha$.

Proof. The second (converse) part of the theorem can be proved in the same way as in [1]. One need only take into account that the implication (FD)→(GrF) holds under assumption (3) on A.

Further, we start from the inclusion $\mathcal{K} \subset \mathcal{M}$. The finite-dimensional C^1-manifold $\mathcal{M}$ in the Banach space X^α is arranged locally as the graph of a smooth function over the tangent subspace. Therefore, for any $w \in \mathcal{M}$ one can find a projector P of rank $n = \dim \mathcal{M}$ and continuous in X^α, a constant $c = c(\mathcal{M}, w, P)$, and a closed ball $\mathcal{V} = \{u \in X^\alpha\colon\ |u - w|_\alpha \leqslant \varepsilon\}$ with $\varepsilon = \varepsilon(w)$ such that $|u - v|_\alpha \leqslant c|P(u - v)|_\alpha$ on $\mathcal{M}(w) = \mathcal{V} \cap \mathcal{M}$. It is obvious that $\mathcal{M}(w)$ is a compact C^1-manifold of dimension n. Using Lemma 2.3 in [1], we choose a projector P_0 of rank n continuous in X^α and $X^{\alpha-1}$ in such a way that

$$|(P - P_0)(u - v)|_\alpha \leqslant (2c)^{-1}|u - v|_\alpha, \qquad u, v \in \mathcal{M}(w).$$

We easily deduce the estimate $|u - v|_\alpha \leqslant 2c|P_0(u - v)|_\alpha$ on $\mathcal{M}(w)$. Hence, (GrL) holds for the invariant compact set $\mathcal{K}$. Theorem 1.4 implies that the phase dynamics on $\mathcal{K}$ is finite-dimensional, which completes the proof.

Remark 1.6. If the limiting dynamics of equation (1) is finite-dimensional, then the periods of its periodic solutions are bounded below by a positive number. This follows from the definition of finite-dimensional dynamics on the attractor and the well-known lower estimate [21] for the periods of periodic solutions of ordinary differential equations in $\mathbb{R}^n$ in terms of the Lipschitz constant of the corresponding vector field.

§ 2. An analytical approach

Our main purpose in this section is to find constructive conditions on the coefficients of equation (1) in the Hilbert space X under which (GrL) holds for the attractor $\mathcal{A} \subset X^\alpha$, which implies that the dynamics on $\mathcal{A}$ is finite-dimensional. The conditions obtained in this section will enable us to prove (in § 3) that the limiting dynamics of parabolic equations (2) is finite-dimensional. Several auxiliary statements can be found in § 4.

Unless otherwise stated, X is assumed to be a Banach space. We use the following notation: $\mathcal{N} = \mathcal{A} \times \mathcal{A}$, D is Fréchet differentiation, $\mathcal{L}(X^\theta, X^\beta)$ is the space of continuous linear operators acting from X^θ to X^β, $\mathcal{L}(X^\theta) = \mathcal{L}(X^\theta, X^\theta)$, $\|\cdot\|$ and $\|\cdot\|_\alpha$ are the norms in $\mathcal{L}(X)$ and $\mathcal{L}(X^\alpha)$, and $\|\cdot\|_{\alpha,0}$ and $\|\cdot\|_{0,\alpha}$ are the norms in $\mathcal{L}(X^\alpha, X)$ and $\mathcal{L}(X, X^\alpha)$. We shall study the (operator) vector fields $\Pi(u, v)$ on $\mathcal{N}$ with values in various Banach spaces Y. We equip $\mathcal{N}$ with the metric induced from $X^\alpha \times X^\alpha$.

Definition 2.1. A continuous field $\Pi\colon \mathcal{N} \to Y$ is said to be *regular* if for any $u, v \in \mathcal{A}$ the function $\Pi(\Phi_t u, \Phi_t v)\colon [0, \infty) \to Y$ belongs to the class C^1 and its derivative $\partial_t \Pi(u, v)$ at zero is bounded uniformly with respect to u and v.

Since the semiflow $\{\Phi_t\}$ is smooth and the compact set $\mathcal{A} \subset X^\alpha$ is invariant, the identical embedding $\mathcal{N} \to X^\alpha \times X^\alpha$ is regular. Hence, any field $\Pi\colon \mathcal{N} \to Y$ extendable to a C^1-map defined on an $(X^\alpha \times X^\alpha)$-neighbourhood of $\mathcal{N}$ is regular. We have

$$\partial_t \Pi(u, v) = D\Pi(u, v)(\partial_t u, \partial_t v), \qquad \partial_t u = G(u) = F(u) - Au.$$

We proceed according to the plan described in the Introduction. For $u, v \in \mathcal{A}$ we put

$$T(u, v) = T_0(u, v) + \int_0^1 DF(\tau u + (1 - \tau)v)\, d\tau, \tag{4a}$$

$$B(u, v) = \omega I + A - T(u, v), \tag{4b}$$

where T_0 is an arbitrary operator field bounded as a map with values in $\mathcal{L}(X^\alpha)$ and regular as a map with values in $\mathcal{L}(X^\alpha, X)$, and $\omega > 0$ is a numerical parameter. Here $T = T_0 + T_1$, $T_1(u,u) = DF(u)$. Since $F \in C^2(X^\alpha, X)$, we have $T_1(u,v) \in \mathcal{L}(X^\alpha, X)$. The field T_1 can be extended from $\mathcal{N}$ to a C^1-map $X^\alpha \times X^\alpha \to \mathcal{L}(X^\alpha, X)$, which implies that the field $T\colon \mathcal{N} \to \mathcal{L}(X^\alpha, X)$ is regular. The operators $T_0(u,v)$ in (4a) play the role of an artificial correction that "improves" the properties of the field $B(u,v)$. Note that

$$\mathcal{D}(B(u,v)) \equiv \mathcal{D}(A) = X^1, \qquad B(u,v) \in \mathcal{L}(X^1, X).$$

The integral mean-value theorem implies that

$$G(u) - G(v) = \bigl(B_0(u,v) - B(u,v)\bigr)(u-v), \qquad B_0(u,v) = \omega I - T_0(u,v).$$

Let

$$\Sigma = \bigcup_{u,v\in\mathcal{A}} \sigma(B(u,v)), \qquad \mathcal{R} = \mathbb{C} \setminus \Sigma,$$
$$\Gamma_a = \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda = a\}, \qquad \Gamma(a,\xi) = \{\lambda \in \mathbb{C}\colon a - \xi \leqslant \operatorname{Re}\lambda \leqslant a + \xi\}$$

with $a > \xi > 0$. We choose ω in (4b) in such a way that (see Lemma 4.1 below) $\operatorname{Re}\Sigma > 0$ for $\Sigma = \Sigma(\omega)$. If $\Gamma_a \subset \mathcal{R}$, then we denote by $P_a(u,v)$ the spectral projector of the operator $B(u,v)$ (continuous in X) corresponding to the part of the spectrum with $\operatorname{Re}\lambda < a$ and put $Q_a(u,v) = I - P_a(u,v)$. According to [2], §§ 1.4, 1.5, the operators $B(u,v)$ are discrete and sectorial in X, $\mathcal{D}(B^\alpha) = X^\alpha$, and the powers B^α and $B^{-\alpha}$ commute with P_a, Q_a and $R(\lambda; B)$. The projectors $P_a(u,v)$ have finite rank $n = n(a)$ for all $u, v \in \mathcal{A}$. Moreover, $P_a, Q_a \in \mathcal{L}(X^\alpha)$, and Lemmas 4.2 and 4.3 below imply that the operator fields $P_a, Q_a\colon \mathcal{N} \to \mathcal{L}(X^\alpha)$, $B^\alpha\colon \mathcal{N} \to \mathcal{L}(X^\alpha, X)$ and $B^{-\alpha}\colon \mathcal{N} \to \mathcal{L}(X, X^\alpha)$ are regular. Since P_a is regular, we have the estimate

$$\|\partial_t P_a(u,v)\|_\alpha \leqslant m(a) \tag{5}$$

on $\mathcal{N}$ with $m(a) < \infty$.

Definition 2.2. When X is a Hilbert space we say that $B\colon \mathcal{N} \to \mathcal{L}(X^1, X)$ is a field of *uniformly scalar type* if $B(u,v) = S^{-1}(u,v)H(u,v)S(u,v)$ on $\mathcal{N}$, where the linear operators $H(u,v)$ are normal in X, the field $S\colon \mathcal{N} \to \mathcal{L}(X)$ is regular, and the field $S^{-1}\colon \mathcal{N} \to \mathcal{L}(X)$ is bounded.

In this definition the $B(u,v)$ are operators of scalar type (see [19]) for all $u, v \in \mathcal{A}$. In the case when $T_0 = 0$ and $u = v$ the representation $B = S^{-1}HS$ in Definition 2.2 was actually used by Kamaev [15], [16] in his study of phase dynamics near the attractors of (scalar or vector) equations of a somewhat wider class than (2). In these papers either the conventional Liouville transformation or [16] a modification of it was applied to the right-hand side of the linearized equation.

The following inequalities should be mentioned in the context of Definition 2.2:

$$\|S(u,v)\| \leqslant \gamma, \qquad \|S^{-1}(u,v)\| \leqslant \gamma, \qquad \|\partial_t S(u,v)\| \leqslant \gamma_1 \tag{6}$$

on $\mathcal{N}$, where $\gamma, \gamma_1 = \text{const}$. It is clear that $\sigma(B) = \sigma(H)$. $P'_a = SP_aS^{-1}$ and $Q'_a = SQ_aS^{-1}$ are orthogonal spectral projectors of the normal operators H. Let us emphasize that the domain of $\mathcal{D}(H) = S\mathcal{D}(B) = SX^1$ depends, generally speaking, on u and v.

If X is a Hilbert space with scalar product $(\,\cdot\,,\cdot\,)$ and there is an $a > \omega$ such that the straight line Γ_a is contained in $\mathcal{R}$, then we put

$$p = S(u,v)B^\alpha(u,v)P_a(u,v)h, \qquad q = S(u,v)B^\alpha(u,v)Q_a(u,v)h,$$
$$W_a(u,v) = \frac{1}{2}\big(|q|^2 - |p|^2\big)$$

for $u, v \in \mathcal{A}$, $h = u - v$. The vector fields $p(u,v)$ and $q(u,v)$ are regular maps to X (as combinations of regular fields). Therefore, the numerical field $W_a(u,v)$ is regular. The following assumption on the dynamics of equation (1) will play an important role:

$$\partial_t W_a(u,v) + 2(a-\omega)W_a(u,v) \leqslant 0 \tag{7}$$

for $u, v \in \mathcal{A}$. Since $Q'_aS = SQ_a$, $Q_aB^\alpha = B^\alpha Q_a$ and $Q_a^2 = Q_a$, we have

$$Q'_a q = Q'_a S B^\alpha Q_a h = SB^\alpha Q_a h = q.$$

We prove likewise that $P'_a p = p$. Since P'_a and Q'_a are orthogonal projectors, we have $(p,q) = 0$. Relation (7) is a non-linear analogue of a similar assumption in [13], Theorem 5, which enables us to construct an inertial manifold for equation (1) with a self-adjoint linear part. If $|q| \geqslant |p|$ (or $|q| \leqslant |p|$), then inequality (7) provides a non-linear generalization of the "squeezing property" and of the "cone condition" [10], Ch. 8, which are well known in the theory of evolutionary systems.

Let L and N be the constants (depending only on A, F and T_0) in Lemmas 4.1 and 4.3 below, and assume that $\|T_0(u,v)\|_\alpha \leqslant K$ on $\mathcal{N}$.

Theorem 2.3. *Let X be a Hilbert space, assume that* (3) *holds for the operator A, and let $\operatorname{Re}\Sigma > 0$ for $\Sigma = \Sigma(\omega)$. Assume that*

(a) *$B\colon \mathcal{N} \to \mathcal{L}(X^1, X)$ is an operator field of uniformly scalar type,*

(b) *$\mathcal{R}$ contains a strip $\Gamma(a,\xi)$ with $a > \omega$ and $\xi \geqslant \gamma\gamma_1 + \gamma^2 N + \gamma^2 L^2(K + m(a))$.*

Then (7) *holds and the limiting dynamics of equation* (1) *is finite-dimensional.*

Note that in the estimate for ξ only m depends on a. The next lemma plays the key role in the proof of the theorem.

Lemma 2.4. *The assumptions of Theorem* 2.3 *imply that inequality* (7) *holds.*

Proof. We do not indicate in our notation that vector fields on $\mathcal{N}$ depend on $u, v \in \mathcal{A}$ and, as a rule, on a. For example, $P_a(u,v) = P$ and $Q_a(u,v) = Q$. For regular operator fields on $\mathcal{N}$ we put for brevity $\partial_t S(u,v) = S_t$ and so on. If $h = u - v$, then $\partial_t h = \partial_t u - \partial_t v = \omega h - T_0 h - Bh$. Here $h \in X^1$. Since $\partial_t h = G(u) - G(v) \in X^\alpha$ and $T_0 h \in X^\alpha$, we have $Bh \in X^\alpha$. Since $X^\alpha = \mathcal{D}(B^\alpha)$, we have[2] $h \in \mathcal{D}(B^{1+\alpha})$ and $B^\alpha h \in X^1$.

[2] $h \in \mathcal{D}(B^2)$ if $T_0 \in \mathcal{L}(X^1)$, in particular, if $T_0 = 0$.

We start from the expression $\partial_t W_a = (\partial_t q, q) - (\partial_t p, p)$ with $q = SB^\alpha Qh$, $p = SB^\alpha Ph$ and $p + q = SB^\alpha h$. First we transform $(\partial_t q, q)$. We put $V = SB^\alpha Q$ and $U = B^{-\alpha} S^{-1}$. Then $q = Vh$, $h = U(p+q)$ and $\partial_t q = V_t h + V\partial_t h$ or, in more detail,

$$\partial_t q = (V_t U + \omega VU - VT_0 U - VBU)(p+q).$$

We shall use the equalities $QB^\alpha = B^\alpha Q$, $QB^{-\alpha} = B^{-\alpha} Q$, $B = S^{-1}HS$ and $Q = S^{-1}Q'S$, as well as $Q'p = 0$ and $Q'q = q$. We see that $SB^\alpha h \in SX^1 = \mathcal{D}(H)$, whence $q = SB^\alpha Qh = Q'SB^\alpha h \in \mathcal{D}(H)$ and $p = SB^\alpha h - q \in \mathcal{D}(H)$. We easily find that $VU(p+q) = q$, $VBU(p+q) = Hq$ and $VT_0U = Q'U_1$ with $U_1 = SB^\alpha T_0 B^{-\alpha} S^{-1}$. Further, we have $Q_t + P_t = 0$ and

$$V_t = S_t B^\alpha Q + S(B^\alpha)_t Q - SB^\alpha P_t = J_1 + J_2 - J_3.$$

Elementary calculations yield the formulae $J_1 U = V_1 Q'$, where $V_1 = S_t S^{-1}$, $J_2 U = V_2 Q'$ with $V_2 = S(B^\alpha)_t B^{-\alpha} S^{-1}$ and $J_3 U = V_3$ with $V_3 = SB^\alpha P_t B^{-\alpha} S^{-1}$. Therefore, $(J_1 U)(p+q) = V_1 q$, $(J_2 U)(p+q) = V_2 q$ and

$$(\partial_t q, q) = ((V_1 + V_2)q, q) - (V_3(p+q), q) - (U_1(p+q), q) + ((\omega I - H)q, q),$$

using the fact that $(Q')^* = Q'$ in X, whence $(Q'U_1(p+q), q) = (U_1(p+q), q)$.

We obtain likewise that

$$(\partial_t p, p) = ((V_1 + V_2)p, p) + (V_3(p+q), p) - (U_1(p+q), p) + ((\omega I - H)p, p).$$

Inequalities (6) and (22) (see the Appendix) imply that $\|V_1\| \leqslant \gamma\gamma_1$ and $\|V_2\| \leqslant \gamma^2 N$. Moreover, $\|V_3\| \leqslant \gamma^2 \|B^\alpha P_t B^{-\alpha}\|$. The obvious identities

$$\|B^\alpha P_t B^{-\alpha}\| = \|A^{-\alpha} B^\alpha P_t B^{-\alpha} A^\alpha\|_\alpha, \qquad \|A^{-\alpha} B^\alpha\|_\alpha = \|B^\alpha A^{-\alpha}\|,$$
$$\|B^{-\alpha} A^\alpha\|_\alpha = \|A^\alpha B^{-\alpha}\|,$$

combined with (5), (6) and (18) imply that $\|V_3\| \leqslant \varkappa = \gamma^2 L^2 m(a)$. As mentioned above, $(p, q) = 0$, whence

$$\bigl|(V_3(p+q), p+q)\bigr| \leqslant \varkappa\bigl(|p|^2 + |q|^2\bigr).$$

The same technique yields the estimate $\|U_1\| \leqslant \varkappa_0 = \gamma^2 L^2 K$, whence

$$\bigl|(U_1(p+q), p) - (U_1(p+q), q)\bigr| \leqslant \varkappa_0 |p+q| \cdot |p-q| = \varkappa_0\bigl(|p|^2 + |q|^2\bigr).$$

Finally, $(Hq, q) \geqslant (a+\xi)|q|^2$ and $(Hp, p) \leqslant (a-\xi)|p|^2$, since the operator H is normal, $q \in Q'X$ and $p \in P'X$. Putting $\varkappa_1 = \gamma\gamma_1 + \gamma^2 N$ and $\varkappa_2 = \varkappa + \varkappa_0$, we obtain that

$$\begin{aligned}\partial_t W_a + 2(a-\omega)W_a &= (\partial_t q, q) - (\partial_t p, p) + (a-\omega)\bigl(|q|^2 - |p|^2\bigr)\\ &\leqslant \varkappa_1|q|^2 + \varkappa_2\bigl(|p|^2 + |q|^2\bigr) - \xi|q|^2 + \varkappa_1|p|^2 - \xi|p|^2\\ &= (\varkappa_1 + \varkappa_2 - \xi)\bigl(|p|^2 + |q|^2\bigr) \leqslant 0,\end{aligned}$$

since $\xi \geqslant \varkappa_1 + \varkappa_2$ by the assumption of the lemma, which completes the proof.

Sometimes it is possible to simplify the restriction on ξ in Theorem 2.3.

Remark 2.5. If $\alpha = 0$, $T_0 = 0$, and the operators $B(u, v)$ are normal, then $\xi \geqslant m(a)$ in assumption (b) of Theorem 2.3.

Here the constants γ_1, K and N are equal to zero, and γ and L are equal to 1. A similar situation arises for the reaction-diffusion equation $u_t = \Delta u + f(x, u)$ in finite domains in $\mathbb{R}^l$, $l \geqslant 1$.

Lemma 2.6. *Let X be a reflexive space, let $\mathcal{K}$ be an invariant compact set of equation* (1) *in X^α, and assume that* (3) *holds for the operator A. If for some $a > 0$ we have $\Gamma_a \subset \mathcal{R}$ and*

$$|Q_a(u,v)(u-v)|_\alpha \leqslant c|P_a(u,v)(u-v)|_\alpha \tag{8}$$

for $u, v \in \mathcal{K}$ and $c = c(\mathcal{K}, a)$, then the phase dynamics on $\mathcal{K}$ is finite-dimensional.

Proof. Put $P_a(w) = P_a(w,w)$ for $w \in \mathcal{K} \subset \mathcal{A}$. By Lemma 4.2, the projector field $P_a \colon \mathcal{N} \to \mathcal{L}(X^\alpha)$ is continuous in the $(X^\alpha \times X^\alpha)$-metric. For every $w \in \mathcal{K}$ we consider an X^α-neighbourhood $\mathcal{V} \supset w$ such that

$$\|P_a(w) - P_a(u,v)\|_\alpha \leqslant (2+2c)^{-1}$$

on $\mathcal{V} \cap \mathcal{K}$. Since $h = u - v$, we have

$$\begin{aligned}|h|_\alpha &\leqslant |P_a(u,v)h|_\alpha + |Q_a(u,v)h|_\alpha \\ &\leqslant (1+c)|P_a(u,v)h|_\alpha \leqslant \frac{1}{2}|h|_\alpha + (1+c)|P_a(w)h|_\alpha,\end{aligned}$$

whence $|u-v|_\alpha \leqslant (2+2c)|P_a(w)(u-v)|_\alpha$ for $u, v \in \mathcal{V} \cap \mathcal{K}$. Hence, (GrL) holds for $\mathcal{K}$. By Theorem 1.4, the dynamics on $\mathcal{K}$ is finite-dimensional, which completes the proof of the lemma.

So, we have obtained another (sufficient) condition for the dynamics of (1) to be finite-dimensional on invariant compact sets.

We are now ready to complete the proof of Theorem 2.3. Let us recall that in many cases we do not indicate in our notation the dependence on $u, v \in \mathcal{A}$ and $a > \omega$. Since the attractor $\mathcal{A}$ is an invariant set, every solution $u(t)$ of equation (1) with $u(0) = u_0 \in \mathcal{A}$ can be extended (as a function of t) to $\mathbb{R}$, and $u(t) \in \mathcal{A}$. We do not affirm yet that this extension is unique for $t < 0$. The identities $B^\alpha Q = QB^\alpha$, $SQ = Q'S$ and the boundedness of $\mathcal{A}$ in X^α imply that

$$2W_a(u,v) \leqslant |q|^2 = |SB^\alpha Qh|^2 = |Q'SB^\alpha h|^2 \leqslant \gamma^2 L^2 |h|_\alpha^2 \leqslant \text{const}$$

for $u, v \in \mathcal{A}$ and $h = u - v$. Here we have used the fact that Q' is an orthogonal projector in X. We have also used inequalities (6) and (18). Now if $u_0, v_0 \in \mathcal{A}$, $\zeta(t) = W_a(u(t), v(t))$ and $\lambda = 2(a-\omega) > 0$, then (7) implies that $\zeta(0) \leqslant e^{\lambda t}\zeta(t)$, $t < 0$. Since $\zeta(t) \leqslant \text{const}$, we have $\zeta(0) \leqslant 0$, that is, $W_a(u,v) \leqslant 0$ on $\mathcal{N}$, or $|q| \leqslant |p|$ with $q = SB^\alpha Qh$, $p = SB^\alpha Ph$. Taking into account (6) and (18), we obtain that

$$|p| \leqslant \gamma |B^\alpha Ph| \leqslant \gamma L |Ph|_\alpha.$$

On the other hand,

$$|Qh|_\alpha = |A^\alpha Qh| = |A^\alpha B^{-\alpha} S^{-1} q| \leqslant \gamma L |q|.$$

Hence, estimate (8) holds with $c = \gamma^2 L^2$. By Lemma 2.6, the phase dynamics on $\mathcal{A}$ is finite-dimensional, which completes the proof of Theorem 2.3.

Remark 2.7. In the Hilbert case the relation (7) implies the estimate (8) and the inequality $W_a(u,v) \leqslant 0$ on $\mathcal{N}$.

Let us see how assumption (b) of Theorem 2.3 is related to the geometry of the total (with respect to $u, v \in \mathcal{A}$) spectrum Σ of the operators $B(u,v)$. We shall discuss the situation when $\mathcal{R} = \mathbb{C} \setminus \Sigma$ contains vertical strips $\Gamma(a,\xi)$ with a and ξ as large as desired. We assume that a suitable choice of the number $\omega > 0$ in (4b) and the parameters $k > 0$ and $0 \leqslant \theta \leqslant 1$ enables one to localize Σ in the domain

$$\Omega(k,\theta) = \bigl\{x + iy \in \mathbb{C}\colon x > 0,\ |y| < kx^\theta\bigr\}.$$

By Lemma 4.1, this is always possible if $\theta = 1$. For $\theta < 1$ such a localization of Σ is typical for semilinear parabolic partial differential equations in finite domains in $\mathbb{R}^l$.

Assumption (a) in Theorem 2.3 implies that

$$\|R(\lambda;B)\| \leqslant \gamma^2/r(\lambda), \qquad B = B(u,v),$$

where $r(\lambda)$ is the distance[3] from $\lambda \in \mathcal{R}$ to Σ and γ is the constant that occurs in (6). Indeed, $R(\lambda;B) = S^{-1}R(\lambda;H)S$ and $\|R(\lambda;H)\| \leqslant 1/r(\lambda)$, since the operators $H = H(u,v)$ are normal. For $u,v \in \mathcal{A}$ and $\Gamma_a \subset \mathcal{R}$, Lemma 4.2 provides the representation

$$\partial_t P_a(u,v) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} E(\lambda;u,v)\,d\lambda \tag{9}$$

with kernel $E(\lambda;u,v) \in \mathcal{L}(X^\alpha)$ holomorphic in $\lambda \in \mathcal{R}$ and such that

$$\|E(\lambda;u,v)\|_\alpha \leqslant \frac{M}{r^2(\lambda)}\bigl(|\lambda|^\alpha + r^\alpha(\lambda)\bigr) \tag{10}$$

with $M = M(A,F,T_0)$.

Using the inclusions $\Sigma \subset \Omega(k,\theta)$ and $\mathcal{R} \supset \Gamma(a,\xi)$ for some $a > \xi > \omega$, we estimate $m(a)$ in (5) in terms of a, ξ and θ. Let $\chi(y)$ be a positive minorant of the function $r(a+iy)$ on $\mathbb{R}$ such that $\chi(y) \leqslant |a+iy|$. Then for $\lambda = a+iy$ the right-hand side of (10) is majorized by the expression $2M|a+iy|^\alpha/\chi^2(y)$. If, moreover, $\chi(y) = \chi(-y)$, then

$$m(a) \leqslant \frac{2M}{\pi}\int_0^\infty \frac{|a+iy|^\alpha}{\chi^2(y)}\,dy. \tag{11}$$

We denote by $c, c_1, \dots$ positive constants depending only on k, θ and M. Increasing, if necessary, the numbers ω and k, we obtain that $\Sigma \subset \Omega(k,\theta) \cap \Omega(k,1)$. We assume that a and ξ are as large as is needed in our arguments below.

We begin our analysis of the right-hand side of (11) with the case $\theta = 0$. We put $\chi(y) = \xi$ on $[0,k]$, $\chi(y) = (\xi^2 + (y-k)^2)^{1/2}$ on (k,∞) and $\chi(y) = \chi(-y)$ for $y < 0$. It is clear that $\chi(y) \leqslant r(a+iy)$ and $\chi(y) \leqslant |a+iy|$ for all $y \in \mathbb{R}$. Using the estimates $|a+iy|^\alpha \leqslant a^\alpha + k^\alpha$ for $0 \leqslant y \leqslant k$, $|a+iy|^\alpha \leqslant a^\alpha + k^\alpha + (y-k)^\alpha$ for $y > k$ and integrating, we obtain the inequality

$$m(a) \leqslant ca^\alpha/\xi.$$

[3] It is easy to show that Σ is closed, whence $r(\lambda) > 0$.

Further, let $\theta = 1$. We put $\chi(y) = \xi$ on $[0, 2ka]$, $\chi(y) = (y - ka)/(1 + k^2)^{1/2}$ on $(2ka, \infty)$ and $\chi(y) = \chi(-y)$ for $y < 0$. If $y > 2ka$, then $\chi(y)$ is the distance from $a + iy$ to the boundary of the sector $\Omega(k, 1)$. Therefore, $\chi(y) \leqslant r(a + iy)$ and $\chi(y) \leqslant |a + iy|$ on $\mathbb{R}$. Writing

$$|a + iy|^\alpha \leqslant a^\alpha(1 + (2k)^\alpha)$$

if $0 \leqslant y \leqslant 2ka$ and

$$|a + iy|^\alpha \leqslant a^\alpha + (ka)^\alpha + (y - ka)^\alpha$$

if $y > 2ka$ and integrating in (11), we obtain the estimate

$$m(a) \leqslant ca^{\alpha+1}/\xi^2.$$

In the case when $0 < \theta < 1$ we put $\nu = \theta^{-1} > 1$ and $\varkappa = k^{-\nu}$. For $x_0 > a$ and $y_0 = kx_0^\theta$ the normal to the parabola $y = kx^\theta$ at (x_0, y_0) intersects the straight line $x = a$ at the point with ordinate

$$y_1 = z(x_0) = y_0 + \nu\varkappa^2 y_0^{2\nu-1} - \nu\varkappa a y_0^{\nu-1},$$

or, which is the same, $z(x_0) = kx_0^\theta + \nu k^{-1}(x_0 - a)x_0^{1-\theta}$. Therefore, $z'(x_0) > 0$ and $y_0 < y_1$. We assume without loss of generality that $y_0 > 1$. If $z(a + \xi) \leqslant 2ka$, then we determine x_0 (and y_0) from the condition $z(x_0) = y_1$, $y_1 = 2ka$. Then $a(2k + cy_0^{\nu-1}) \geqslant c_1 y_0^{2\nu-1}$ and $a \geqslant c_2 y_0^\nu$. If $z(a + \xi) > 2ka$, then we put $y_0 = k(a + \xi)^\theta$ and $y_1 = z(a + \xi)$. In any case, we have $1 < y_0 < y_1$, $y_0 \leqslant c_3 a^\theta$ and $y_1 \geqslant 2ka$. Simple geometrical arguments taking into account that $y = kx^\theta$ is a concave function show that the distance from any point $\lambda = a + iy$, $y_0 \leqslant y \leqslant y_1$, to the branch of the parabola with $x \geqslant a + \xi$ exceeds $\rho(y) = (\xi^2 + (y - y_0)^2)^{1/2}$. Hence, this is true for $0 \leqslant x < a - \xi$.

We put $\chi(y) = \xi$ on $[0, y_0]$, $\chi(y) = \rho(y)$ on $(y_0, y_1]$, $\chi(y) = (y - ka)/(1 + k^2)^{1/2}$ if $y > y_1$ and $\chi(y) = \chi(-y)$ if $y < 0$. We see that $\chi(y) \leqslant r(a + iy)$ if $y_0 < y \leqslant y_1$. This estimate is obvious on $[0, y_0]$. For $y > y_1$ it has already been established.

On the other hand, $\chi(y) \leqslant |a + iy|$ on $\mathbb{R}$. We also have $|a + iy|^\alpha \leqslant a^\alpha + y_0^\alpha$ for $0 \leqslant y \leqslant y_0$, $|a + iy|^\alpha \leqslant a^\alpha + y_0^\alpha + (y - y_0)^\alpha$ for $y_0 < y \leqslant y_1$ and $|a + iy|^\alpha \leqslant a^\alpha + (ka)^\alpha + (y - ka)^\alpha$ for $y > y_1$. Estimating the right-hand side of (11), we obtain the inequality

$$m(a) \leqslant c_4 a^\alpha/\xi + c_5 a^{\alpha+\theta}/\xi^2.$$

In all three cases we have used the freedom of choice of parameters a and ξ mentioned above.

Hence, the assumption on ξ in Theorem 2.3 for sufficiently large $a > 0$ can be stated as follows: $\xi \geqslant ca^{\alpha/2}$ if $0 \leqslant \theta \leqslant \alpha/2$ and $\xi \geqslant ca^{(\alpha+\theta)/3}$ if $\alpha/2 < \theta \leqslant 1$, where $c = c(A, F, T_0)$. The threshold value $\theta = \alpha/2$ arises from the equality $(\alpha + \theta)/3 = \alpha/2$.

Thus we obtain an important consequence of Theorem 2.3.

Theorem 2.8. *Let X be a Hilbert space, assume that* (3) *holds for the operator A, and let assumption* (a) *of Theorem* 2.3 *hold. Assume, moreover, that the set $\mathcal{R}$ contains strips $\Gamma(a_n, \xi_n)$ with $a_n, \xi_n \to \infty$ as $n \to \infty$ and $\Sigma \subset \Omega(k, \theta) \cap \Omega(k, 1)$ for some $k > 0$ and $\theta \in [0, 1]$. Put $\beta = \alpha/2$ if $0 \leqslant \theta \leqslant \alpha/2$ and $\beta = (\alpha + \theta)/3$ if $\alpha/2 < \theta \leqslant 1$.*

If $a_n^\beta = o(\xi_n)$ as $n \to \infty$, then the dynamics of equation (1) *on the attractor is finite-dimensional and relation* (7) *holds with a suitable $a > 0$.*

Note that $\beta \leqslant (\alpha + 1)/3 < 2/3$. Besides, $\theta = 0$ if $\Sigma \subset \mathbb{R}$.

If A is a positive self-adjoint operator on X, then it is easy to establish the inclusion $\Sigma \subset \Omega(k, \alpha)$ with a suitable $k > 0$ (by varying the parameter ω). Therefore, in Theorem 2.8, $a_n^{2\alpha/3} = o(\xi_n)$ as $n \to \infty$. The well-known spectral jump condition (see [12], [10], [13]), under which equation (1) has an inertial manifold is more restrictive (for $\alpha > 0$), since it can be stated in comparable terms as follows: $a_n^\alpha = o(\xi_n)$ as $n \to \infty$. So the limiting dynamics in the class of evolution systems (1) can be finite-dimensional even if there is no inertial manifold.

Remark 2.9. We can weaken assumption (a) on $B(u, v)$ in Theorem 2.3 by replacing the $\mathcal{L}(X^\alpha)$-boundedness of the correcting field T_0 in (4a) by $\mathcal{L}(X^\alpha, X^{\alpha-\varepsilon})$-boundedness with a sufficiently small $\varepsilon > 0$. The corresponding version of Theorem 2.8 could lead to new applications of the theory developed here and in [1].

§ 3. Partial differential equations

The above results enable us to establish that the limiting dynamics of equations (2) is finite-dimensional. We consider the differential operator $\partial_{xx} h = h_{xx}$ on $L^2(0, 1)$ either with the Sturm boundary conditions

$$h(0)\cos\mu_0 + h_x(0)\sin\mu_0 = 0, \qquad h(1)\cos\mu_1 + h_x(1)\sin\mu_1 = 0, \tag{12a}$$

where $\mu_0, \mu_1 \in (-\pi/2, \pi/2]$, or with periodic solutions

$$h(0) = h(1), \qquad h_x(0) = h_x(1). \tag{12b}$$

Information concerning the spaces of differentiable functions used below can be found in [22]–[24].

Let $\mathfrak{I} = [0, 1]$ in case (12a) and let $\mathfrak{I}$ be the circle of circumference 1 in case (12b). We denote by $\mathcal{H}^s = \mathcal{H}^s(\mathfrak{I})$ the generalized Sobolev L^2-spaces with arbitrary $s \geqslant 0$. Note that the space $\mathcal{H}^s$ with $s > 1/2$ is a Banach algebra [22], 2.8.3. The operator $u \to u_x$ is a continuous map from $\mathcal{H}^{s+1}$ to $\mathcal{H}^s$. Embedding theorems imply that if s, $\nu \geqslant 1$ are integers and $g\colon \mathfrak{I} \times \mathbb{R}^2 \to \mathbb{R}$ is a smooth function, then the map $u \to g(x, u, u_x)$ belongs to the class $BC^\nu(\mathcal{H}^{s+1}, \mathcal{H}^s)$ if $g \in C^{s+\nu}$ and to the class[4] $BC(\mathcal{H}^{s+1}, \mathcal{H}^s)$ if $g \in C^s$.

The assumption on $f(x, u, p)\colon \mathfrak{I} \times \mathbb{R}^2 \to \mathbb{R}$ in (2) can be stated as follows.

Assumption 3.1. *The function f belongs to C^3. If in case* (12a) $\mu_j = 0$ *for $j = 0$ or $j = 1$, then $f(j, 0, p) = 0$ for all $p \in \mathbb{R}$.*

We reduce equation (2) to (1) with $X = L^2(\mathfrak{I})$. The linear operator ∂_{xx} is self-adjoint on X. If $\varkappa \geqslant 0$ is suitably chosen, then the operator $A = \varkappa I - \partial_{xx}$

[4]The classes of maps BC^ν and BC are defined in § 1.

is positive definite [25], Ch. 1, and discrete. Hence, it satisfies hypothesis (H1) and generates a Hilbert semiscale $\{X^\alpha\}_{\alpha\geqslant 0}$. It is well known [23], Ch. 5, that X^α are closed subspaces (with equivalent norms) in $\mathcal{H}^{2\alpha}$, and $X^\alpha = \mathcal{H}^{2\alpha}$ in case (12b). The latter is also true for the boundary conditions (12a) for $\alpha \leqslant 1/4$ (for $\alpha \leqslant 3/4$ if $\mu_0, \mu_1 \neq 0$). If $\alpha > 3/4$, then we have continuous embeddings $X^\alpha \subset C^1(\mathfrak{I})$ and $X^{\alpha+1/2} \subset C^2(\mathfrak{I})$. The embedding $C(\mathfrak{I}) \subset X$ is also continuous. This implies, in particular, that $F \in BC^3(X^\alpha, X)$ for the map $F\colon u \to \varkappa u + f(x,u,u_x)$. If for some $\alpha \in (3/4, 1)$ equation (2) is dissipative in X^α, then hypotheses (H1)–(H3) hold for its abstract form $\partial_t u = -Au + F(u)$. Hence, the constructions of §§ 1, 2 are valid. Assumption 3.1 on the non-linear part of (2) (redundant in comparison with (H2)) implies supplementary qualities of phase dynamics. Let $\mathcal{A}$ be the attractor and $\{\Phi_t\}$ the dissipative semiflow of (2) in X^α, let $\mathcal{N} = \mathcal{A} \times \mathcal{A}$, and let Y be a Banach space.

Remark 3.2. If assumptions 3.1 and (H3) hold with $\alpha \in (3/4, 1)$, then equation (2) has the following properties:

(a) the attractor $\mathcal{A}$ is bounded in $X^{3/2}$ (or X^2) if assumption (12a) (or (12b)) holds,

(b) every field $\Pi\colon \mathcal{N} \to Y$ continuous in the $(X^\alpha \times X^\alpha)$-metric that can be extended to a C^1-map $X^1 \times X^1 \to Y$ is regular.

Indeed, taking into account the relation between X^s and $\mathcal{H}^{2s}$, we deduce from 3.1 that $F \in BC^2(X^1, X^{1/2})$. For the boundary conditions (12b) we have, moreover, $F \in BC^1(X^{3/2}, X^1)$. Remarks 1.2 and 1.3 enable us to establish that $\mathcal{A}$ is compact in X^1 and the map $(t,u) \to \Phi_t u\colon (0,\infty) \times X^1 \to X^1$ is smooth. Hence, the identical embedding $\mathcal{N} \to X^1 \times X^1$ is regular (see Definition 2.1), which implies that (b) holds. Finally, (a) follows directly from the remarks cited.

Conditions for the X^α-dissipativity of (2) can be obtained on the basis of well-known a priori estimates [26], [9], [27] for solutions of these equations using abstract methods of functional analysis [2], [9], [28]. For example, if $f(x,u,p)$ satisfies assumption 3.1 and, moreover, satisfies the following conditions [9], Ch. 1, § 7: $f(x,u,0)\operatorname{sign} u \to -\infty$ as $|u| \to \infty$ uniformly with respect to $x \in \mathfrak{I}$ and

$$|f| + |f_x| + |f_u| \leqslant M(u)(1+p^2), \qquad |f_p| \leqslant M(u)(1+|p|),$$

then problem (2), (12a) with $\mu_0 = \mu_1 = 0$ $(3/4 < \alpha < 1)$ and the periodic problem (2), (12b) are X^α-dissipative. Indeed, for every $u_0 \in X^\alpha \subset C^1(\mathfrak{I})$ equation (2) has a local solution $u(t) \in C([0,t^*), X^\alpha)$, $u(t) \in X^1$ on $(0,t^*)$. If $u_0 \in X^1$, then $u(t) \in C([0,t^*), X^1)$ (see Remark 1.2). In any case $u(t) \in X^{3/2}$ for $t \in (0,t^*)$, that is, $u(t)$ belongs to the Hölder class $C^{2+\delta}(\mathfrak{I})$ with some $0 < \delta < 1$, and Theorems 1.7.2 and 2.5.1 in [9] are applicable. Although these theorems were proved for Dirichlet boundary conditions, they can be proved in the periodic situation in a similar way. One way or another, we establish that equation (2) has a solution (global with respect to $t > 0$)

$$u(t) = \Phi_t u_0 \in X^\alpha \cap C^{2+\delta}(\mathfrak{I}), \qquad u_0 \in X^\alpha,$$

the phase semiflow $\{\Phi_t\}$ in X^α is $(C(\mathfrak{I}), C(\mathfrak{I}))$-bounded uniformly with respect to $t \geqslant 0$, and there is an invariant compact set $\mathcal{A} \subset X^\alpha$ that is a $(C(\mathfrak{I}),$

$C^{2+\delta}(\mathfrak{I})$)-attractor. (We retain here the convenient terminology used in [9].) Since the embeddings $X^\alpha \subset C(\mathfrak{I})$ and $C^{2+\delta}(\mathfrak{I}) \subset \mathcal{H}^2$ are continuous, $\mathcal{A}$ is an (X^α, X^1)-attractor ($\mathcal{A}$ attracts balls $\mathcal{B} \subset X^\alpha$ uniformly in the norm of X^1 as $t \to +\infty$), which implies that (2) is dissipative in X^α.

Let us explain why we cannot construct an inertial manifold for problems (2), (12). If we number the eigenvalues λ_n of A in increasing order and put $a_n = (\lambda_{n+1}+\lambda_n)/2$, $\xi_n = (\lambda_{n+1}-\lambda_n)/2$ for $n \geqslant 1$, then [25] $a_n \sim cn^2$ and $\xi_n \sim cn$ as $n \to \infty$, $c = \text{const}$. The spectral jump condition $a_n^\alpha = o(\xi_n)$, sufficient for the phase dynamics to be "asymptotically finite-dimensional", would imply the inequality $\alpha < 1/2$, which is impossible even under the most stringent conditions on the dependence on u_x of the non-linearity of f.

However, the limiting dynamics of equation (2) is finite-dimensional.

Theorem 3.3. *Assume that* 3.1 *holds for* $f\colon \mathfrak{I} \times \mathbb{R}^2 \to \mathbb{R}$, *and let* $\alpha \in (3/4, 1)$. *If equation* (2) *with one of the boundary conditions* (12) *is dissipative in* X^α, *then its phase dynamics on the attractor is finite-dimensional.*

Proof. This reduces to the verification of the assumptions of Theorem 2.8. Relation (3) obviously holds for the positive-definite operator A. If $u, h \in X^\alpha$, then

$$DF(u)h = \varkappa h + f_u(x, u, u_x)h + f_p(x, u, u_x)h_x.$$

Put

$$b(x; u, v) = \int_0^1 f_p(x, w, w_x)\, d\tau, \qquad b_0(x; u, v) = \int_0^1 f_u(x, w, w_x)\, d\tau \tag{13}$$

for $u, v \in X^\alpha$ and $w = \tau u + (1-\tau)v$. It is convenient to treat these expressions as Bochner integrals with values in suitable function spaces, which enables us to analyze them using the corresponding technique [29], Ch. 3. Since $f \in C^3$, we have $f_p, f_u \in C^2$, and the maps $\Pi_1\colon u \to f_p(x, u, u_x)$ and $\Pi_2\colon u \to f_u(x, u, u_x)$ belong to $BC(\mathcal{H}^3, \mathcal{H}^2)$. By Remark 3.2, (a), the convex hull $\mathcal{A}^c$ of the attractor $\mathcal{A}$ is bounded in $X^{3/2} \subset \mathcal{H}^3$. Hence, the sets $\Pi_1\mathcal{A}^c$ and $\Pi_2\mathcal{A}^c$ are bounded in $\mathcal{H}^2$. Therefore, the functions b, b_0, and b^2 are bounded in the norms of the Banach algebras $\mathcal{H}^2$ and $C^1(\mathfrak{I})$ uniformly with respect to $(u, v) \in \mathcal{N}$. In the case of boundary conditions (12b) we likewise obtain that these functions are uniformly $\mathcal{H}^3$-bounded using the fact that $\mathcal{A}$ is X^2-bounded.

Formatting our formulae as we did in (4), we write

$$T(u, v)h = T_0(u, v)h + \varkappa h + b_0(x; u, v)h + b(x; u, v)h_x, \tag{14a}$$
$$B(u, v) = (\omega + \varkappa)I - \partial_{xx} - T(u, v) \tag{14b}$$

for $u, v \in \mathcal{A}$. Let us recall that the operator field T_0 on $\mathcal{N}$ must be bounded as a map with values in $\mathcal{L}(X^\alpha)$ and regular as a map with values in $\mathcal{L}(X^\alpha, X)$. First we put $T_0 = 0$ and choose a number $\omega > 0$ using Lemma 4.1. We denote by $S(u, v)$ the operator of multiplication by a positive function $\psi(x; u, v) \in C^2[0, 1]$ such that $(\ln\psi)_x = b/2$ and $\psi\big|_{x=0} = 1$ (this operator is continuous in $X = L^2(\mathfrak{I})$). In the case of periodic boundary conditions we have, generally speaking, $\psi\big|_{x=1} \neq 1$ and

$\psi \notin C(\mathfrak{I})$. The transformation $\eta = S(u,v)h$ for $h \in X^1$ enables us to write the operators $B = B(u,v)$ as $B = S^{-1}HS$, where $H = H(u,v)$,

$$H(u,v)\eta = \omega\eta - \eta_{xx} - q(x;u,v)\eta \tag{15}$$

and $q = b_0 - b^2/4 - b_x/2$. The functions $q \in \mathcal{H}^1 \subset C(\mathfrak{I})$ are bounded in the norm of $\mathcal{H}^1$ uniformly with respect to $u, v \in \mathcal{A}$. The domain $\mathcal{D}(H)$ of operators H coincides with $S\mathcal{D}(B)$, where $\mathcal{D}(B) = \mathcal{D}(A) = X^1$, that is, the boundary conditions can change when we pass from B to H. Nevertheless, the type of (12a) is, on the whole, preserved, and the operators $H(u,v)$ turn out to be sef-adjoint in X. Conditions (12b) are transformed into $\eta(1) = \rho\eta(0)$ and $\eta_x(1) = \rho\eta_x(0)$ with $\rho = \psi\big|_{x=1}$ (since the functions f_p and b are periodic in x). Here the $H(u,v)$ with $\rho \neq 1$ are not even normal. In both cases we have[5] $\|S\| = |\psi|_C$, and that of the operator field $S\colon \mathcal{N} \to \mathcal{L}(X)$ follows from that of the function field $\psi(\,\cdot\,;u,v)\colon \mathcal{N} \to C[0,1]$, which, in turn, follows from that of the function field $b(\,\cdot\,;u,v)\colon \mathcal{N} \to C(\mathfrak{I})$. Using the fact that the non-linear operator $(u,v) \to f_p(x,w,w_x)\colon X^\alpha \times X^\alpha \to C(\mathfrak{I})$ with $w = \tau u + (1-\tau)v$, $\tau \in [0,1]$, is C^1-smooth and differentiating the expressions in (13) with respect to the parameter $(u,v) \in X^\alpha \times X^\alpha$ under the integrals, we establish that the map $\Pi\colon (u,v) \to b(\,\cdot\,;u,v)$ belongs to the class $C^1(X^\alpha \times X^\alpha, C(\mathfrak{I}))$. Hence, its restriction to $\mathcal{N}$ is regular. We prove likewise that the function field $b_0\colon \mathcal{N} \to C(\mathfrak{I})$ in (13) is regular. We deduce the regularity of the field $b^2\colon \mathcal{N} \to C(\mathfrak{I})$ from that of b using the multiplicative structure of $C(\mathfrak{I})$.

Hence, in the case of the Sturm conditions (12a) the operators $H(u,v)$ are self-adjoint, the field S is regular, and $\|S^{-1}\| = |\psi^{-1}|_C \leqslant \text{const}$ on $\mathcal{N}$. Therefore, $B\colon \mathcal{N} \to \mathcal{L}(X^1, X)$ is an operator field of uniformly scalar type in the sense of Definition 2.2. Since $\Sigma = \Sigma(B) \subset \mathbb{R}^+$, the assumption $\Sigma \subset \Omega(k,\theta) \cap \Omega(k,1)$ of Theorem 2.8 obviously holds with $\theta = 0$ and any $k > 0$.

Let us establish that Σ is sufficiently rarefied. Using the asymptotics [25], Ch. 1, of the eigenvalues of the operators (15), we see that (in terms of Theorem 2.8) we can put $a_n = \pi^2 n^2 + cn$ and $\xi_n = \pi^2 n + c_1$ with $n \geqslant n_0$, where c, c_1 and n_0 depend on μ_0, μ_1 and the majorant of $\mathcal{H}^1$-norms of the functions $q(\,\cdot\,;u,v)$. In the same terms we have $\beta = \alpha/2 < 1/2$ and $a_n^\beta = o(\xi_n)$ as $n \to \infty$, which completes the proof of Theorem 3.3 for problem (2), (12a).

Now let us consider the periodic conditions (12b). We put $T_0(u,v)h = -q(x;u,v)h$ in (14a). The operator field $B(u,v)$ in (14b) changes accordingly. In this situation $X^\alpha = \mathcal{H}^{2\alpha}$ is a Banach algebra and, as mentioned above, the functions b, b_0 and b^2 are bounded in the norm of $\mathcal{H}^3$ uniformly with respect to $(u,v) \in \mathcal{N}$. Therefore, $|q(\,\cdot\,;u,v)|_1 \leqslant \text{const}$. Consequently, $|q(\,\cdot\,;u,v)|_\alpha \leqslant \text{const}$. Hence, the multipliers $T_0(u,v)$ belong to $\mathcal{L}(X^\alpha)$ and $\|T_0(u,v)\|_\alpha \leqslant \text{const}$ for $u, v \in \mathcal{A}$. Formulae (14b) and (15) imply that $B = S^{-1}H_0S$, where $S(u,v)$ are the operators defined above and $H_0 = H_0(u,v) = \omega I - \partial_{xx}$ with the boundary conditions $h(1) = \rho h(0)$, $h_x(1) = \rho h_x(0)$ and $\rho = \rho(u,v) = \psi(x;u,v)\big|_{x=1} > 0$. It is easy to calculate the eigenvalues λ and the eigenfunctions $\chi(x)$ of the operator $(-\partial_{xx})$:

$$\begin{aligned} &\lambda_0 = -\ln^2 p, \qquad &&\lambda_{n,1} = (2\pi n - i\ln\rho)^2, \qquad &&\lambda_{n,2} = (2\pi n + i\ln\rho)^2, \\ &\chi_0 = \rho^x, &&\chi_{n,1} = \rho^x e^{2\pi n i x}, &&\chi_{n,2} = \rho^x e^{-2\pi n i x} \end{aligned} \tag{16}$$

[5] $|\cdot|_C$ is the norm in $C[0,1]$.

for $n \geqslant 1$. The system of functions $\{\chi_0, \chi_{n,1}, \chi_{n,2}\}$ is complete and orthogonal in $L^2(\mathfrak{I})$ with the weight ρ^{-2x}. Hence, $H_0 = S_0^{-1} H_1 S_0$, where the operators $H_1 = H_1(u,v)$ are normal in X, $\mathcal{D}(H_1) = \mathcal{D}(A) = X^1$ and $S_0(u,v)h = \rho^{-x}h$ for $h \in X$. We see that $B = S_1^{-1} H_1 S_1$ with $S_1 = S_0 S$ and $\|S_1^{-1}\| \leqslant \|S^{-1}\| \cdot \|S_0^{-1}\| \leqslant \text{const}$ on $\mathcal{N}$.

In the context of the assumptions of Theorem 2.8 and Definition 2.2, the operator fields $S_1 \colon \mathcal{N} \to \mathcal{L}(X)$ and $T_0 \colon \mathcal{N} \to \mathcal{L}(X^\alpha, X)$ should be regular. The above-proved regularity of the fields ψ and S implies that the fields $S_0, S_1 \colon \mathcal{N} \to \mathcal{L}(X)$ are regular. Since the embedding $X^\alpha \subset C(\mathfrak{I})$ is continuous, $\|T_0\|_{\alpha,0} \leqslant c|q|$ (here $|\cdot|$ is the norm in X and the constant c does not depend on q) and the field T_0 is regular if the function field $q \colon \mathcal{N} \to X$, $q = b_0 - b^2/4 - b_x/2$, is regular. The function fields b_0 and b^2 on $\mathcal{N}$ are regular as maps with values in $C(\mathfrak{I})$. Hence, they are regular as maps with values in X.

Let $\Pi_\tau(u,v) = (f_p(x,w,w_x))_x$ with $w = \tau u + (1-\tau)v$, $\tau \in [0,1]$ and arbitrary $u = u(x)$, $v = v(x) \in X^1$, and let $\Pi(u,v)$ be the result of integrating $\Pi_\tau(u,v)$ with respect to τ. It is clear that $\Pi(u,v) = (b(x;u,v))_x$ for $u, v \in \mathcal{A} \subset X^1$. By Remark 3.2, (b) the regularity of the function field $b_x \colon \mathcal{N} \to X$ will follow from the inclusions $\Pi\big|_{\mathcal{N}} \in C(\mathcal{N}, X)$ and $\Pi \in C^1(X^1 \times X^1, X)$. Since $f \in C^3$ and $f_p \in C^2$, the map $u \to f_p(x,u,u_x)$ belongs to the class $BC^1(X^1, X^{1/2})$, whence $\Pi_\tau \in C^1(X^1 \times X^1, X)$. Differentiating the integral expression for Π with respect to the parameter $(u,v) \in X^1 \times X^1$, we obtain that $\Pi \in C^1(X^1 \times X^1, X)$. Further, the operators $u \to g(x,u,u_x)$, $g = f_{px}$, f_{pu}, f_{pp}, act continuously from X^α to $C(\mathfrak{I})$ and $(f_p(x,u,u_x))_x = f_{px} + f_{pu}u_x + f_{pp}u_{xx}$ for $u \in \mathcal{A}^c \subset X^1$. By Lemma 1.1, the function $u \to Au \colon \mathcal{A} \to X$, $Au = \varkappa u - u_{xx}$, is continuous in the X^α-metric. The same is true for the maps $u \to u_{xx}$ and $u \to u_x$ from $\mathcal{A}^c$ to X. Hence, $\Pi_\tau, \Pi\big|_{\mathcal{N}} \in C(\mathcal{N}, X)$, the fields b_x, q and T_0 are regular, and $B(u,v)$ is an operator field of uniformly scalar type on $\mathcal{N}$.

It remains to specify the value of ω in (14b) and to find sufficiently wide lacunae in $\Sigma = \Sigma(B) \subset \mathbb{C} = \{x+iy\}$. Let $\omega > \ln^2 \rho(u,v) + 1$ on $\mathcal{N}$. Then (16) implies that $\Sigma \subset \Omega(k,\theta)$, $\theta = 1/2$, $k = 2(\omega-1)^{1/2}$. In fact Σ is contained in the domain $|y| \leqslant k(x-1)^\theta$, whence $\Sigma \subset \Omega(k_1, 1)$ with $k_1 = k_1(\omega) > 0$. Formatting our statements as in Theorem 2.8, we put $a_n = 4\pi^2(n^2+n+1/2)$ and $\xi_n = 2\pi^2(n+1/2)$ for $n > \omega/2\pi^2$. Moreover, $\theta > \alpha/2$ and $\beta = (\alpha+\theta)/3 < 1/2$, since $\alpha < 1$. Therefore, $a_n^\beta = o(\xi_n)$ as $n \to \infty$. Hence, the limiting dynamics for problem (2), (12b) is finite-dimensional, which completes the proof of Theorem 3.3.

Note that Theorem 2.8 implies that (7) holds for the dynamics of (2) on the attractor $\mathcal{A}$. The structure of W_a and the value of a in (7) depend on the choice of T_0 and ω in (14).

For example, the reaction-diffusion equation with non-linear convection,

$$u_t = u_{xx} + (g(x,u))_x + g_0(x,u), \qquad x \in (0,1),$$

with standard conditions at $x = 0, 1$ has finite-dimensional limiting dynamics. Assumptions on the smooth functions g and g_0 are determined by Assumption 3.1 and the dissipativity of the equation under investigation in X^α with $\alpha \in (3/4, 1)$.

Remark 3.4. Theorem 3.3 can be generalized to one-dimensional systems

$$u_t^j = (d(x)u_x^j)_x + f_j(x,u,u_x), \qquad 1 \leqslant j \leqslant l,$$

where $u = (u^1, u^2, \dots, u^l)$, with the Dirichlet boundary condition (the coefficient $d(x) > 0$ is smooth on $[0,1]$) if we use the above-mentioned analogue of the Liouville transformation [16].

Apparently, it can be proved that semilinear parabolic equations of order greater than two on $(0,1)$ have finite-dimensional limiting dynamics if the boundary conditions are not too pathological and the non-linear part satisfies appropriate conditions. However, the proof of such statements would involve a certain modification of the constructions used in § 2 (see Remark 2.9) and in this section.

§ 4. Appendix

In this appendix we collect technical statements concerning the properties of the derivative of the vector field $F(u) - Au$ of equation (1) in the Banach space X. As before, we start from the basic hypotheses (H1)–(H3). We write $\lambda = x + iy$ for $\lambda \in \mathbb{C}$. Let us recall that $\mathcal{A}$ is the attractor of (1) in X^α, $\mathcal{N} = \mathcal{A} \times \mathcal{A}$, the spectrum $\sigma(A)$ is contained in the sector $\Omega = \{\lambda\colon\ |y| < kx\}$ with $k > 0$ and the estimate $\|R(\lambda; A)\| \leqslant M/(1 + |\lambda|)$ for the resolvent $R(\lambda; A) = (A - \lambda I)^{-1}$ holds in $\Omega_1 = \mathbb{C} \setminus \Omega$. Our notation for spaces and norms of linear operators corresponds to that assumed in § 2. Operator fields T and B on $\mathcal{N}$ are defined by formulae (4a) and (4b) with $\omega \geqslant \omega_0$, where ω_0 is the constant that occurs in the next lemma. The field $T\colon \mathcal{N} \to \mathcal{L}(X^\alpha, X)$ is regular in the sense of Definition 2.1.

Lemma 4.1. *For $\omega \geqslant \omega_0 > 0$ and $B = B(u,v)$ with $u, v \in \mathcal{A}$, the spectrum $\sigma(B)$ is contained in Ω,*

$$\|R(\lambda; B)\| \leqslant \frac{M_1}{1 + |\lambda|} \tag{17}$$

if $\lambda \in \Omega_1$, and

$$\|A^\alpha B^{-\alpha}\| \leqslant L, \qquad \|B^\alpha A^{-\alpha}\| \leqslant L. \tag{18}$$

The constants M_1 and L depend only on A. The constant ω_0 depends only on A, F and T_0, where T_0 is the field of operators in (4a).

We shall use the moment inequality [2], Theorem 1.4.4,

$$\|A^\alpha V\| \leqslant \Theta \|AV\|^\alpha \|V\|^{1-\alpha} \tag{19}$$

for the sectorial operator A with V, $AV \in \mathcal{L}(X)$ and $\Theta = \Theta(k, M)$. For the rest of this appendix, $c, c_1, \dots$ are constants depending only on A, F and T_0. We identify the dependence of any objects on the operator A with their dependence on the parameters (k, M), although these parameters cannot be determined unambiguously from A.

Proof. Since the operator field $T = T(u,v)$ is regular, we have $\|TA^{-\alpha}\| \leqslant c$ on $\mathcal{N}$. Let $R = R(\lambda; A)$ for $\lambda \in \Omega_1$. Then (formally) $R(\lambda; A - T) = R(I - TR)^{-1}$ and $\lambda \notin \sigma(A - T)$ if $\|TR\| < 1$. It is clear that

$$\|TR\| = \|TA^{-\alpha} A^\alpha R\| \leqslant c \|A^\alpha R\|.$$

Since $AR = I + \lambda R \in \mathcal{L}(X)$, formula (19) implies that

$$\|A^\alpha R\| \leqslant \Theta \|AR\|^\alpha \|R\|^{1-\alpha}.$$

Hence,

$$\|TR\| \leqslant c\Theta\big(\|R\|^{1-\alpha} + |\lambda|^{\alpha}\|R\|\big).$$

Using the estimate for $R(\lambda; A)$, we obtain that

$$\|TR\| \leqslant c_1(1+|\lambda|)^{\alpha-1}, \qquad c_1 = c\Theta(M^{1-\alpha} + M).$$

Hence, $\|TR\| \leqslant 1/2$ if $|\lambda|^{1-\alpha} \geqslant 2c_1$, and geometrical arguments show that for all $u, v \in \mathcal{A}$ the spectrum $\sigma(A - T(u,v))$ is contained in the sector $|y| < k(x+\omega_0)$ with $\omega_0 = \rho\varkappa$, $\rho^{1-\alpha} = 2c_1$ and $\varkappa = (1+k^{-2})^{1/2}$, where $\omega_0 = \omega_0(k, M, T)$, that is, ω_0 in fact depends on A, F and T_0. Outside the specified sector we have $\|TR\| \leqslant 1/2$ and $\|(I - TR)^{-1}\| \leqslant (1 - \|TR\|)^{-1} \leqslant 2$, whence

$$\|R(\lambda; A - T)\| \leqslant 2\|R(\lambda; A)\| \leqslant 2M/(1+|\lambda|).$$

If $\omega \geqslant \omega_0$ in (4b), then $\sigma(B(u,v)) \subset \Omega$ and $\|R(\lambda; B)\| \leqslant 2M/(1+|\lambda-\omega|)$ on Ω_1. Solving an elementary extremum problem, we obtain that

$$|\lambda| \leqslant \varkappa|\lambda - \omega|, \qquad \|R(\lambda; B)\| \leqslant M_1(1+|\lambda|)$$

with $M_1 = 2\varkappa M$ if $\lambda \in \Omega_1$. The operators $A^{\alpha}B^{-\alpha}$ and $B^{\alpha}A^{-\alpha}$ are bounded in X by Theorem 1.4.6 in [2]. A careful analysis of the corresponding calculations in [2] shows that the norms of these operators can be estimated in terms of k, M and $M_1 = M_1(k, M)$, which completes the proof of the lemma.

Let us recall that $r(\lambda)$ is the distance from $\lambda \in \mathbb{C}$ to the total spectrum $\Sigma = \Sigma(B)$, $\mathcal{R} = \mathbb{C} \setminus \Sigma$, and Γ_a is the straight line $x = a$ in $\mathbb{C}$. We have just proved that $\Sigma \subset \Omega$ and $\mathcal{R} \supset \Omega_1$. The projectors P_a were defined in § 2.

Lemma 4.2. *If $\Gamma_a \subset \mathcal{R}$ for some $a > 0$ and*

$$\|R(\lambda; B)\| \leqslant \frac{c}{r(\lambda)} \quad \text{on} \quad \mathcal{R} \tag{20}$$

for $B = B(u,v)$, $u, v \in \mathcal{A}$, then the projector field $P_a \colon \mathcal{N} \to \mathcal{L}(X^{\alpha})$ is regular and the integral representation (9) *holds for $\partial_t P_a(u,v)$, where the kernel $E(\lambda; u, v) \in \mathcal{L}(X^{\alpha})$ is holomorphic in $\lambda \in \mathcal{R}$ and satisfies estimate* (10).

Proof. Consider the triangular positively oriented contour $\Gamma \subset \mathcal{R}$ in $\mathbb{C}$ with vertices $(0,0)$, $(a, -ka)$ and (a, ka), where k is the parameter of the sectorial operator A. Let $(u,v) \in \mathcal{N}$, $T = T(u,v)$ and $R = R(\lambda; B) = R(\lambda; u, v)$ for $\lambda \in \mathcal{R}$. Let us use the Riesz formula

$$P_a(u,v) = -\frac{1}{2\pi i}\int_{\Gamma} R(\lambda; u, v)\, d\lambda. \tag{21}$$

Since $BR = I + \lambda R$, we have $BR \in \mathcal{L}(X)$, $B^{\alpha}R \in \mathcal{L}(X)$ and $R \in \mathcal{L}(X, X^{\alpha})$. Taking into account (18), we obtain that

$$\|R\|_{0,\alpha} = \|A^{\alpha}R\| = \|A^{\alpha}B^{-\alpha}B^{\alpha}R\| \leqslant L\|B^{\alpha}R\|.$$

Putting $V = R$ in (19) and replacing A by the sectorial operator B, we deduce that

$$\|B^\alpha R\| \leqslant \Theta\bigl(\|R\|^{1-\alpha} + |\lambda|^\alpha \|R\|\bigr).$$

Lemma 4.1 implies that $\Theta = \Theta(k, M_1) = \Theta(A)$.

Formula (20) shows that

$$\|R\|_{0,\alpha} \leqslant c_1\bigl(\|R\|^{1-\alpha} + |\lambda|^\alpha \|R\|\bigr) \leqslant K(\Lambda)$$

on $\Lambda \times \mathcal{N}$ for an arbitrary closed set $\Lambda \subset \mathcal{R}$. Since the field $T\colon \mathcal{N} \to \mathcal{L}(X^\alpha, X)$ is regular, the second resolvent identity [24], 3.2.1, enables us to establish that the operator field $R\colon \mathcal{N} \to \mathcal{L}(X, X^\alpha)$ is regular for every $\lambda \in \Lambda$. We establish likewise that $\partial_t R = R\partial_t T R$ and the inequality

$$\|\partial_t R\|_{0,\alpha} \leqslant c_2\bigl(\|R\|^{1-\alpha} + |\lambda|^\alpha \|R\|\bigr)^2 \leqslant K_1(\Lambda)$$

holds. Let us note that for $\Lambda = (-\infty, 0]$ the arguments used in this paragraph remain valid if we replace assumption (20) by estimate (17).

We have established that $\|R(\lambda; u, v)\|_{0,\alpha} \leqslant \text{const}$ on $\Gamma \times \mathcal{N}$. The above-cited resolvent identity enables us to deduce the continuity of the field $P_a\colon \mathcal{N} \to \mathcal{L}(X, X^\alpha)$ from that of the field $R\colon \mathcal{N} \to \mathcal{L}(X, X^\alpha)$ with $\lambda \in \Gamma$. The function $\partial_t T(\Phi_t u, \Phi_t v)\colon (u, v, t) \to \mathcal{L}(X^\alpha, X)$ is bounded on $\mathcal{N} \times [0, \infty)$ and continuous with respect to $t \geqslant 0$. Therefore, the function $\partial_t R(\lambda; \Phi_t u, \Phi_t v)\colon (\lambda, u, v, t) \to \mathcal{L}(X, X^\alpha)$ is bounded on $\Gamma \times \mathcal{N} \times [0, \infty)$ and continuous with respect to $t \geqslant 0$. Therefore, we can differentiate the integral (21) with respect to t along the solutions of equation (1) and the projector field P_a is regular as a map with values in $\mathcal{L}(X, X^\alpha)$. Hence, this field is regular as a map with values in $\mathcal{L}(X^\alpha)$. For the derivative $\partial_t P_a(u, v)$ at zero we obtain an expression similar to (9) with kernel

$$E(\lambda) = E(\lambda; u, v) = -\partial_t R(\lambda; u, v) \in \mathcal{L}(X^\alpha)$$

holomorphic on $\mathcal{R}$ and contour of integration Γ. Formula (18) and the identity $RB^\alpha = B^\alpha R$ imply that

$$\|R\|_\alpha = \|A^\alpha B^{-\alpha} R B^\alpha A^{-\alpha}\| \leqslant L^2 \|R\|.$$

Since $\|\partial_t T\|_{\alpha,0} \leqslant c_3$, we have

$$\|E(\lambda)\|_\alpha \leqslant c_3 \|R\|_{0,\alpha} \|R\|_\alpha \leqslant c_4\bigl(\|R\|^{2-\alpha} + |\lambda|^\alpha \|R\|^2\bigr)$$

on $\mathcal{R}$, and the desired estimate (10) follows from inequality (20). Inequality (17) implies that $\|R(\lambda; B)\| = O(|\lambda|^{-1})$ as $\lambda \to \infty$ in Ω_1. Therefore, $\|E(\lambda)\|_\alpha = O(|\lambda|^{\alpha-2})$, $\alpha < 1$. Taking into account that $E(\lambda)$ is holomorphic, we obtain the representation (9) for $\partial_t P_a(u, v)$ with contour of integration Γ_a, which completes the proof of the lemma.

Lemma 4.3. *Let $B = B(u,v)$, $u, v \in \mathcal{A}$. Then the operator fields $B^{-\alpha}\colon \mathcal{N} \to \mathcal{L}(X, X^\alpha)$ and $B^\alpha\colon \mathcal{N} \to \mathcal{L}(X^\alpha, X)$ are regular. The estimate*

$$\|(\partial_t B^\alpha)B^{-\alpha}\| \leqslant N \tag{22}$$

holds on $\mathcal{N}$ with a constant $N = N(A, F, T_0)$.

Proof. We can assume without loss of generality that $\alpha > 0$. It is well known [2], § 1.4, that

$$B^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_{-\infty}^{0} |\lambda|^{-\alpha} R(\lambda; B)\, d\lambda \tag{23}$$

for $B = B(u,v)$. If $\lambda \leqslant 0$, then the intermediate results obtained in the preceding proof imply that the field $R(\lambda; B)\colon \mathcal{N} \to \mathcal{L}(X, X^\alpha)$ is regular and $\partial_t R = R\partial_t TR$. Using formula (17), we write the estimates for R and $\partial_t R$ obtained there as

$$\|R\|_{0,\alpha} \leqslant c_1(1+|\lambda|)^{\alpha-1}, \qquad \|\partial_t R\|_{0,\alpha} \leqslant c_2(1+|\lambda|)^{2\alpha-2}.$$

Let us emphasize that all this is true without condition (20) on $R(\lambda; B)$. The second resolvent identity enables us to deduce the continuity of the field $B^{-\alpha}\colon \mathcal{N} \to \mathcal{L}(X, X^\alpha)$ from that of the field $R\colon \mathcal{N} \to \mathcal{L}(X, X^\alpha)$ with $\lambda \leqslant 0$. The function $\psi(\lambda) = |\lambda|^{-\alpha}(1+|\lambda|)^{2\alpha-2}$ is integrable on $(-\infty, 0)$. Applying Lebesgue's theorem on passage to the limit under the integral and using the above arguments, we see that it is possible to differentiate expression (23) with respect to t along the solutions of equation (1) and the field $B^{-\alpha}$ is regular. Using (18), we have

$$\|B^\alpha \partial_t R\| = \|B^\alpha A^{-\alpha} A^\alpha \partial_t R\| \leqslant L\|\partial_t R\|_{0,\alpha} \leqslant c_3(1+|\lambda|)^{2\alpha-2}.$$

Formula (23) implies that $\|B^\alpha \partial_t B^{-\alpha}\| \leqslant N$ for $u, v \in \mathcal{A}$ with $N = N(A, F, T_0)$.

It remains to observe that $B^\alpha B^{-\alpha} = I$ and $\|B^\alpha\|_{\alpha,0} = \|B^\alpha A^{-\alpha}\| \leqslant L$. The regularity of the field B^α can be deduced from that of $B^{-\alpha}$ by obvious operator transformations. We have $(\partial_t B^\alpha)B^{-\alpha} + B^\alpha \partial_t B^{-\alpha} = 0$, whence $\|(\partial_t B^\alpha)B^{-\alpha}\| = \|B^\alpha \partial_t B^{-\alpha}\| \leqslant N$, which completes the proof of the lemma.

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