Věta C Existují fce $\sin x$ a $\cos x$ a číslo $\pi > 0$ t. ž.

1. $\sin(x+y) = \sin x \cdot \cos y + \cos x \cdot \sin y$
   $\cos(x+y) = \cos x \cdot \cos y - \sin x \cdot \sin y \quad \forall x, y \in \mathbb{R}$

2. $\sin(-x) = -\sin x$, $\cos(-x) = \cos x \quad \forall x \in \mathbb{R}$

3. $\sin x$, $\cos x$ spojité v $\mathbb{R}$

4. $\sin x$ rostoucí v $[0, \frac{\pi}{2}]$, $\sin 0 = 0$, $\sin \frac{\pi}{2} = 1$

5. $$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Odvození dále plyne:

- $\cos 0 = 1$, neboť

$$1 = \sin\left(\frac{\pi}{2} + 0\right) = \underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1} \cdot \cos 0 + \cos\left(\frac{\pi}{2}\right) \cdot \underbrace{\sin 0}_{=0} = \cos 0$$

- $\cos^2 x + \sin^2 x = 1$, neboť

$$1 = \cos 0 = \cos(x - x) = \cos x \cdot \cos(-x) - \sin x \cdot \sin(-x) = \cos^2 x + \sin^2 x$$

- $|\sin x|, |\cos x| \leq 1, \ \forall x \in \mathbb{R}$, neboť

$$|\sin x| = \sqrt{\sin^2 x} \leq \sqrt{\sin^2 x + \cos^2 x} = \sqrt{1} = 1$$

- $\cos\left(\frac{\pi}{2}\right) = 0$, $\cos \pi = -1$, $\sin\left(-\frac{\pi}{2}\right) = -1$
- $\cos(x+\pi) = -\cos x$, $\sin(x+\pi) = -\sin x$
- $\sin x, \cos x$ jsou 2π-periodické, neboť

$$\sin(x+2\pi) = \sin((x+\pi)+\pi) = -\sin(x+\pi) = -(-\sin x) = \sin x$$

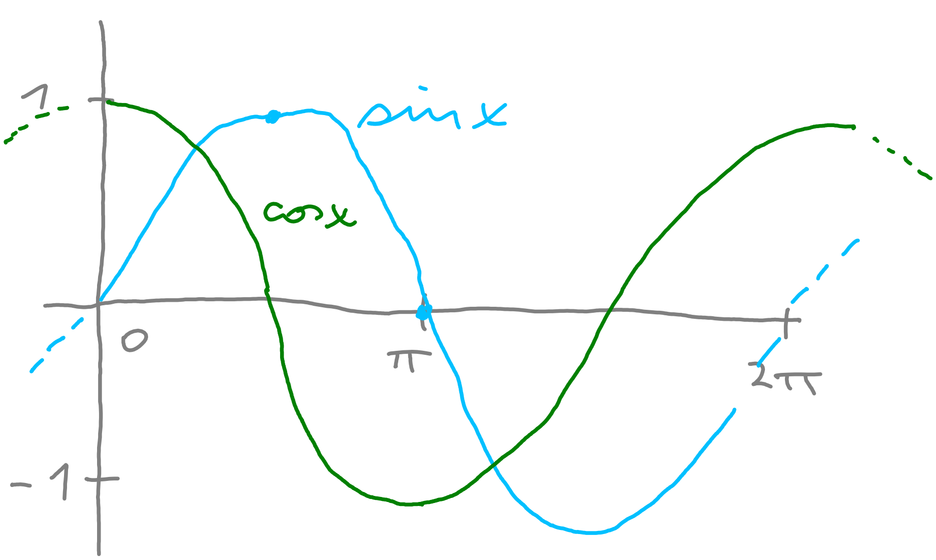
- $\sin x, \cos x$ lze vzájemně nahradit, tj.

$$\sin x = \cos\left(x - \frac{\pi}{2}\right), \quad \cos x = \sin\left(x + \frac{\pi}{2}\right)$$

- $$\sin a - \sin b = 2\cos\left(\frac{a+b}{2}\right) \cdot \sin\left(\frac{a-b}{2}\right)$$

$$\cos a - \cos b = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

TRIK: $x = \frac{a+b}{2}$, $y = \frac{a-b}{2}$, tj. $a = x+y$, $b = x-y$



Další užitečné vzorce:

$$\sin 2x = 2\sin x \cos x \;, \quad \cos 2x = \cos^2 x - \sin^2 x$$

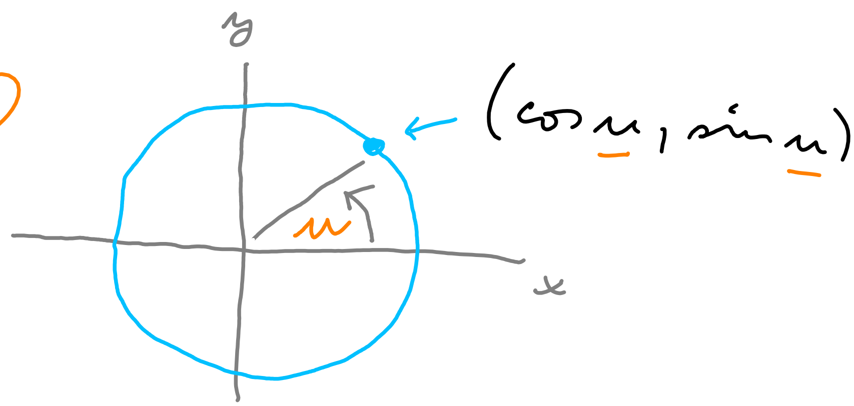
$$\cos^2 x = \frac{1}{2}(1+\cos 2x), \quad \sin^2 x = \frac{1}{2}(1-\cos 2x)$$

$$\sin(x+k\pi) = (-1)^k \sin x \;, \quad \cos(x+k\pi) = (-1)^k \cos x \qquad (k \in \mathbb{Z})$$

$$\sin x = 0 \iff x = k\pi \;, \; k\in\mathbb{Z}$$

$$\cos x = 0 \iff x = \frac{\pi}{2} + k\pi \;, \; k \in \mathbb{Z}$$

Použití: ①



② řešení ODR: $y''(x) + y(x) = 0$

neboť: $(\sin x)'' = (\cos x)' = -\sin x$

$(\cos x)'' = (-\sin x)' = -\cos x$

Věta D Existuje fce $\ln(x)$ v $(0,+\infty)$ t.ž.

1. $\ln(xy) = \ln x + \ln y$, $\forall x,y \in (0,+\infty)$
2. $\ln x$ je rostoucí, spojité na $(0,+\infty)$
3. $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x} = 1$.

Důsledky:

- $\ln 1 = 0$, neboť $\underline{\ln 1} = \ln 1\cdot 1 = \underline{\ln 1} + \underline{\ln 1}$
- $\ln \frac{1}{x} = -\ln x$, $\forall x>0$, neboť $0 = \ln 1 = \ln x\cdot\frac{1}{x}$ $= \ln x + \ln\frac{1}{x}$
- $\ln(x^n) = n\cdot \ln x$, $\forall x>0, n\in\mathbb{N}$

$\ln(\sqrt[k]{x}) = \frac{1}{k}\ln x$, $\forall x>0, k\in\mathbb{N}$

neboť: $\underline{\ln x} = \ln\left(\sqrt[k]{x}\right)^k = \underline{k\cdot\ln\sqrt[k]{x}}$

Limity: $\lim\limits_{y\to 1} \dfrac{\ln y}{y-1} = 1$ $\left(\underset{x\to 0}{1+x} = \underset{y\to 1}{y}\right)$

$\lim\limits_{x\to+\infty} \ln x = +\infty \iff \forall \varepsilon>0\ \exists \delta>0 : x\in P(+\infty,\delta)$

tj. $x > \frac{1}{\delta} \Rightarrow \ln x > \frac{1}{\varepsilon}$ $\Rightarrow \ln x \in U(+\infty,\varepsilon)$

$\varepsilon > 0$ ... dáno: $2 > 1 \Rightarrow \ln 2 > \ln 1 = 0$

$\exists n \in \mathbb{N} : n \cdot \ln 2 > \frac{1}{\varepsilon}$

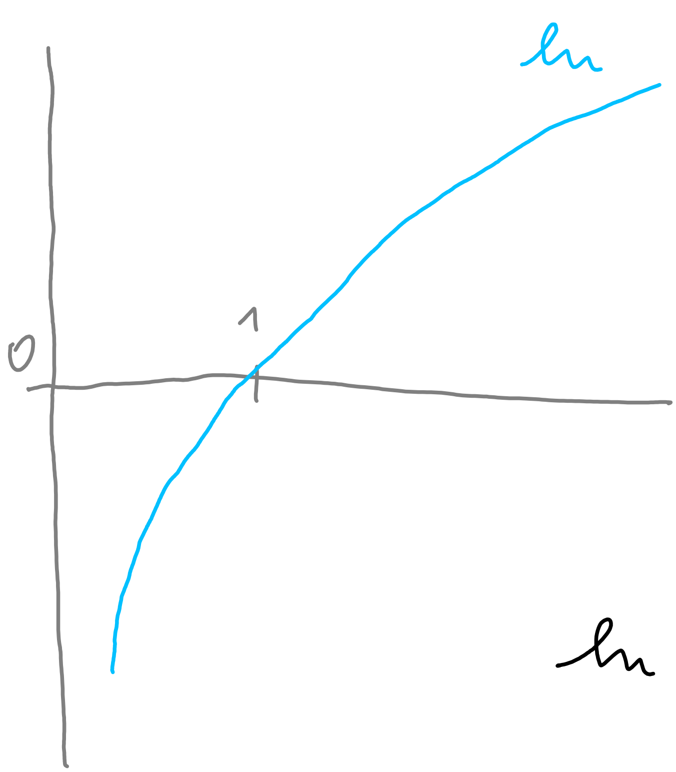
polož $\delta = \frac{1}{2^n}$: $x > \frac{1}{\delta} = 2^n \Rightarrow \ln x > \ln 2^n$

$= n \cdot \ln 2 > \frac{1}{\varepsilon}$

$\Rightarrow \lim_{x \to +\infty} \ln x = +\infty$, neboť:

$\underbrace{\lim_{x \to 0+} \ln x = -\infty}_{\text{L.2.3}}$

$$\lim_{y \to +\infty} \ln\left(\frac{1}{y}\right) = \lim_{y \to +\infty} (-\ln y) = -\infty$$



$\ln((0, +\infty)) = \mathbb{R}$

Růstové škály:

$$\lim_{x\to+\infty} \frac{\ln x}{x^a} = 0 \qquad (a>0)$$

$$\lim_{x\to 0+} x^b \cdot \ln x = 0 \qquad (b>0)$$