③ $\int R(\cos x, \sin x)\,dx$ $\qquad R = R(u,v)$ ... rac. fce

(i) substituce $t = \operatorname{tg}\frac{x}{2}$ $\qquad x \in (-\pi, \pi) \longleftrightarrow t \in \mathbb{R}$

$$\boxed{x = 2\operatorname{arctg} t = \varphi(t)}$$

Snažíme: $\rightsquigarrow$ racionální fce v t

$$dx = \varphi'(t)\,dt = \frac{2}{1+t^2}\,dt$$

$$\cos x = \cos 2\cdot\frac{x}{2} = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = 2\cos^2\frac{x}{2} - 1 = \frac{2}{1+\operatorname{tg}^2\frac{x}{2}} - 1$$

$$= \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2}$$

$$\sin x = \sin 2\frac{x}{2} = \frac{2\sin\frac{x}{2}\cos^2\frac{x}{2}}{\cos\frac{x}{2}} = \frac{2t}{1+t^2}$$

vzorečky: $\cos 2y = 2\cos^2 y - 1$

$\sin 2y = 2\sin y \cos y$

$\sin^2 y + \cos^2 y = 1 \quad \Big/ \frac{1}{\cos^2 y} \quad \Rightarrow \quad \operatorname{tg}^2 y + 1 = \frac{1}{\cos^2 y}$

CELKEM:

$$\int R(\cos x, \sin x)\,dx = \int R\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)\frac{2\,dt}{1+t^2}$$

Př. $\displaystyle\int \frac{dx}{2+\cos x} = \left| t = \operatorname{tg}\frac{x}{2} \right| = \int \frac{1}{2+\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt$

$$= \int \frac{2\,dt}{t^2+3} = \frac{2}{3}\int \frac{dt}{\left(\frac{t}{\sqrt{3}}\right)^2+1} = \frac{2}{\sqrt{3}}\operatorname{arctg}\frac{t}{\sqrt{3}}$$

$$\Rightarrow \quad \underbrace{\int \frac{dx}{2+\cos x}}_{f(x)} = \underbrace{\frac{2}{\sqrt{3}} \operatorname{arctg}\left(\frac{\operatorname{tg}\frac{x}{2}}{\sqrt{3}}\right)}_{F(x)} \qquad \boxed{\text{KDE ?}}$$

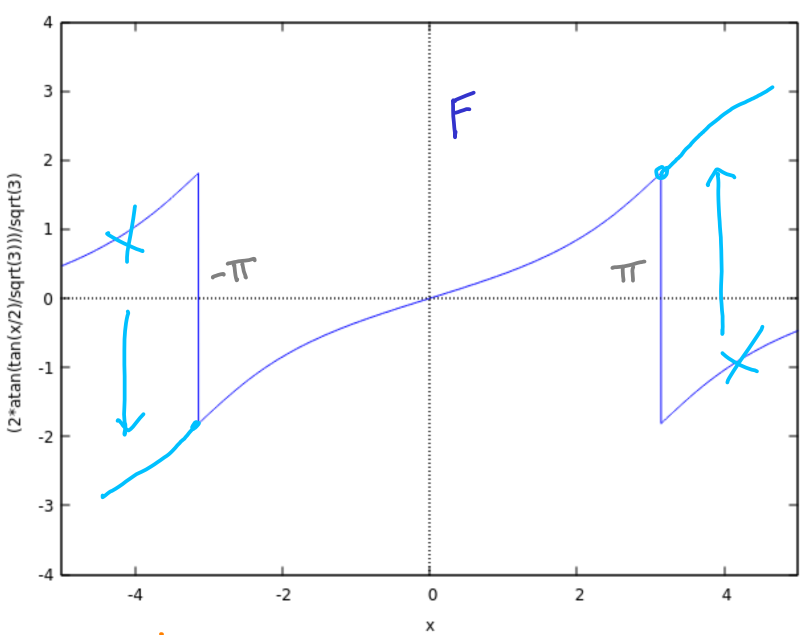
platí: $F'(x) = f(x)$, $x \in (-\pi, \pi)$ ← subst.

$F(x), f(x) \ldots 2\pi$-periodické

$\Rightarrow \quad x \in (\pi, 3\pi), (3\pi, 5\pi), \ldots$

obecně: $\forall x \in ((2k-1)\pi, (2k+1)\pi)$

$k \in \mathbb{Z}$ pevné



napojení v bodě $x = \pi$: zřejmě: $F(x) \to \pm \frac{\pi}{\sqrt{3}}$

$x \to \pi \mp$ (VoLSF)

$$F_1(x) = \begin{cases} \frac{2}{\sqrt{3}} \operatorname{arctg}\left(\frac{\operatorname{tg}\frac{x}{2}}{\sqrt{3}}\right), & x \in (-\pi, \pi) \\ \frac{\pi}{\sqrt{3}} & x = \pi \\ \frac{2}{\sqrt{3}} \operatorname{arctg}\left(\frac{\operatorname{tg}\frac{x}{2}}{\sqrt{3}}\right) + \frac{2\pi}{\sqrt{3}}, & x \in (\pi, 3\pi) \end{cases}$$

vidíme: $F_1(\pi) = \lim_{x \to \pm\pi} F_1(x)$ $\overset{(V.2.5)}{\Rightarrow}$ $F_1(x)$ spojitá v bodě π

$\Rightarrow$ $F_1'(x) = f(x), \ \forall x \in (-\pi, 3\pi)$

dk: $x \neq \pi$ ... výpočet výše

$x = \pi$ $\Leftarrow$ Lemma 6.2

TEDY: $\int f(x)\,dx = F_1(x), \ x \in (-\pi, 3\pi)$

(ii) $t = \operatorname{tg} x$ $\qquad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longleftrightarrow t \in \mathbb{R}$

$x = \operatorname{arctg} t = \varphi(t)$

$dx = \dfrac{dt}{1+t^2}$

$\sin^2 x + \cos^2 x = 1 \ \Big/ \ \dfrac{1}{\cos^2 x}$

$\operatorname{tg}^2 x + 1 = \dfrac{1}{\cos^2 x}$

Podmínka:

$$\boxed{R(u,v) = R(-u,-v)}$$

$\underbrace{|\cos x|}_{\cos x} = \sqrt{\cos^2 x} = \dfrac{1}{\sqrt{1+\operatorname{tg}^2 x}} = \dfrac{1}{\sqrt{1+t^2}}$

$\sin x = \operatorname{tg} x \cdot \cos x = \dfrac{t}{\sqrt{1+t^2}}$

Př. $\displaystyle\int \frac{dx}{1+\sin^2 x} \ \Big| \ t = \operatorname{tg} x \ \Big| = \int \underbrace{\frac{1}{1+\left(\frac{t}{\sqrt{1+t^2}}\right)^2}}_{\frac{1}{2t^2+1}} \cdot \frac{dt}{1+t^2}$

$R(u,v) = \dfrac{1}{1+v^2}$

$$\int \frac{dt}{2t^2+1} = \int \frac{dt}{(\sqrt{2}t)^2+1} = \frac{1}{\sqrt{2}} \operatorname{arctg} \sqrt{2}t, \quad t \in \mathbb{R}$$

$$\Rightarrow \int \frac{dx}{1+\sin^2 x} = \frac{1}{\sqrt{2}} \operatorname{arctg}(\sqrt{2} \cdot \operatorname{tg} x), \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Pozn.:
$$\int \frac{dx}{1+\sin^2 x} \left| \, t = \operatorname{tg}\frac{x}{2} \, \right| = \int \underbrace{\frac{1}{1+\left(\frac{2t}{1+t^2}\right)^2} \cdot \frac{2\,dt}{1+t^2}}_{\frac{2t^2+2}{t^6+6t^2+1}}$$

(iii) $t = \sin x$ rez. $t = \cos x$

$dt = \cos x \, dx$ $\boxed{1.\text{VoS}}$ $dt = -\sin x \, dx$

$\boxed{R(-u,v) = -R(u,v)}$ $\boxed{R(u,-v) = -R(u,v)}$

Př.:
$$\int \frac{\sin x \cdot \cos x}{1+\sin x} dx \,\Big|\, t = \sin x \,\Big| = \int \frac{t}{1+t} dt$$

„dt“

$$R(u,v) = \frac{v \cdot u}{1+v}$$

$$= \int 1 - \frac{1}{1+t}\, dt = t - \ln|1+t|, \quad t \in (-\infty, -1), \ (-1, +\infty)$$

$$\Rightarrow \int \frac{\sin x \cdot \cos x}{1+\sin x}\, dx = \sin x - \ln(1+\sin x)$$

$$x \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right) + 2k\pi$$

Pozn.

$$\int \frac{\sin x \cdot \cos x}{1+\sin x}\, dx = \left| t = \operatorname{tg}\frac{x}{2} \right|$$

$$= \int \underbrace{\frac{\frac{2t}{1+t^2} \cdot \frac{1-t^2}{1+t^2}}{1+\frac{2t}{1+t^2}} \cdot \frac{2t}{1+t^2}}_{\frac{4t(1-t^2)}{(1+t)(1+t^2)^2}}\, dt$$

CELKEM: pořadí preference:

1. $t = \sin x$

   nebo

   $t = \cos x$

2. $t = \operatorname{tg} x$

3. $t = \operatorname{tg}\frac{x}{2}$