Rozklady polynomů:

$$Q(x) = A_0 x^N + A_1 x^{N-1} + \dots$$

$$A_i \in \mathbb{R},\ N = \operatorname{st} Q \in \mathbb{N}$$

$$\Rightarrow \quad Q(x) = A_0 \prod_{j=1}^{r} (x-a_j)^{n_j}$$

$a_j \in \mathbb{C}$ kořeny násobnosti

$$n_j \in \mathbb{N},\ \sum_{j=1}^{r} n_j = N$$

Platí: $a = \alpha + i\beta,\ \beta \neq 0$ ... kořen nás. n

$\Leftrightarrow \bar{a} = \alpha - i\beta$ ... " – " –

$$\Big(\underbrace{(x-a)(x-\bar{a})}_{\parallel}\Big)^n = (x^2+bx+c)^n$$

$$x^2 - (a+\bar{a})x + a\bar{a} = x^2 + bx + c$$

kde $b = -2\alpha,\ c = \alpha^2 + \beta^2$

$$D = b^2 - 4c = -\beta^2 < 0$$

Rozklad – reálné verze:

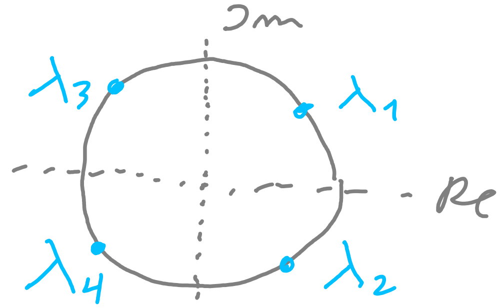
$$Q(x) = A_0 \prod_{j=1}^{m} (x-a_j)^{n_j} \prod_{r=1}^{n} (x^2 + b_r x + c_r)^{q_r} \quad (*)$$

$$a_j \in \mathbb{R},\ b_r, c_r \in \mathbb{R},\ b_r^2 - 4c_r < 0$$

Příkl. $x^4+1 = ?$

kořeny: $\lambda_{1,2} = \frac{1 \pm i}{\sqrt{2}}$

$\lambda_{3,4} = \frac{-1 \pm i}{\sqrt{2}}$



$$\Rightarrow \quad x^4+1 = (x-\lambda_1)(x-\bar{\lambda}_1)\cdot(x-\lambda_3)(x-\bar{\lambda}_3)$$
$$= (x^2-\sqrt{2}x+1)\cdot(x^2+\sqrt{2}x+1)$$

TRIK: pomocí vzorce $A^2-B^2=(A+B)(A-B)$

$$x^4+1 = (x^2+1)^2-2x^2 = (x^2+1)^2-(\sqrt{2}x)^2$$
$$= (x^2+1+\sqrt{2}x)\cdot(x^2+1-\sqrt{2}x)$$

nebo: $y=x^2 \ \dots \ y^2=-1 \Rightarrow y=\pm i$

$\Rightarrow x = \pm\sqrt{\pm i}$

Věta F nechť $R(x) = \frac{P(x)}{Q(x)}$, $P(x), Q(x)$ jsou polynomy, st $P <$ st Q. nechť $Q(x)$ má rozklad (*). Pak $\exists!$ čísla $A_{jr}, B_{rs}, C_{rs} \in \mathbb{R}$ t.ž.

$$R(x) = \sum_{j=1}^{m}\sum_{r=1}^{k_j} \frac{A_{jr}}{(x-a_j)^r} + \sum_{r=1}^{n}\sum_{s=1}^{l_r} \frac{B_{rs}x+C_{rs}}{(x^2+b_r x+c_r)^s}$$

pro x t.ž. $Q(x) \neq 0$.

Integrace rac. fce: obecný postup

1. st $P \geq$ st Q $\Rightarrow$ dělením $R(x) = r(x) + \dfrac{\tilde{P}(x)}{Q(x)}$

   kde st $\tilde{P} <$ st Q

2. $\dfrac{\tilde{P}(x)}{Q(x)}$ rozložím dle Věty F

3. integruji jednotlivé členy rozkladu

$$\int \frac{dx}{(x-a)^n} = \begin{cases} \ln|x-a|, & n=1 \\ \dfrac{1}{(1-n)(x-a)^{n-1}}, & n \geq 2 \end{cases}$$

$$\int \frac{Bx+C}{(x^2+bx+c)^n}\,dx = \frac{B}{2}\underbrace{\int \frac{2x+b}{(x^2+bx+c)^n}}_{I_1} + \left(C - \frac{bB}{2}\right)\underbrace{\int \frac{dx}{(x^2+bx+c)^n}}_{I_2}$$

$$I_1 = \int \frac{f'(x)\,dx}{(f(x))^n} \; \Big| \; 1.\,\mathrm{VoS} \; \Big| = \int \frac{dy}{y^n}$$

$y = f(x) = x^2+bx+c$

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ad I_2: úprava jmenovatele:

$$x^2+bx+c = \left(x+\frac{b}{2}\right)^2 + \underbrace{c - \frac{b^2}{2}}_{= d^2 > 0} = d^2\left[\left(\frac{x}{d}+\frac{b}{2d}\right)^2 + 1\right]$$

$$\Rightarrow \quad I_2 = \left| \begin{array}{l} y = \left(\frac{x}{d} + \frac{b}{2d}\right) \\ dy = \frac{1}{d}\,dx \end{array} \right| = \frac{1}{d}\int \frac{dy}{(y^2+1)^n}$$

... umím (rekurentní vzorec)

Kde? na každém intervalu kde $Q(x) \neq 0$, tj. např. pro I_1 na $(-\infty, a)$, $(a, +\infty)$.

Příkl. $\int \frac{3x}{x^3-1}\,dx = \ln|x-1| - \frac{1}{2}\ln(x^2+x+1) + \sqrt{3}\,\mathrm{arctg}\left(\frac{2x+1}{\sqrt{3}}\right)$

$x \in (-\infty, 1), (1, +\infty)$

$x^3 - 1 = (x-1)(x^2+x+1)$ .... Věta F $\Rightarrow$

$$\boxed{\frac{3x}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}}, \quad x \neq 1$$

$$\Rightarrow \quad 3x = A(x^2+x+1) + (Bx+C)(x-1)$$

potřebuji rovnice pro A, B, C:

- dosazuji $x \in \mathbb{C}$
- koeficienty u x^k, $k = 0, 1, 2, \ldots$

$$\begin{array}{ll} x=1: & 3 = 3A \\ x=0: & 0 = A - C \\ \text{koef. } x^2: & 0 = A + B \end{array} \quad \Rightarrow \quad A = 1,\ B = -1,\ C = 1$$

$$\Rightarrow \int \frac{3x}{x^3-1} = \underbrace{\int \frac{dx}{x-1}}_{\substack{\shortparallel \\ \ln|x-1|}} + \underbrace{\int \frac{(-x+1)}{x^2+x+1}dx}_{\substack{\shortparallel \\ I_2}}$$

$\sim (-\infty,1), (1,+\infty)$

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ad I_2: $$= \underbrace{-\frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx}_{} + \frac{3}{2}\underbrace{\int \frac{dx}{x^2+x+1}}_{I_4}$$

$$\left.\begin{aligned} y &= x^2+x+1 \\ dy &= (2x+1)dx \end{aligned}\right| = \int \frac{dy}{y} = \ln|y|$$

$$= \ln(x^2+x+1),\ x\in\mathbb{R}$$

ad I_4: $$x^2+x+1 = \left(x+\frac{1}{2}\right)^2+\frac{3}{4} = \frac{3}{4}\left(\left(\frac{2x+1}{\sqrt{3}}\right)^2+1\right)$$

subst.: $$y = \frac{2x+1}{\sqrt{3}}$$

$$dy = \frac{2}{\sqrt{3}}dx$$

$$= \frac{2}{\sqrt{3}}\int \frac{dy}{y^2+1}$$

$$= \frac{2}{\sqrt{3}} \operatorname{arctg} y = \frac{2}{\sqrt{3}} \operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right),\ x\in\mathbb{R}$$

$(y\in\mathbb{R})$