1. VoS (Věta 5.3)

$$\int g(f(x))\cdot f'(x)\,dx \;\left|\begin{array}{l} y=f(x) \\ dy=f'(x)\,dx \end{array}\right| = \int g(y)\,dy = G(y)$$

$x\in I$ $\qquad$ $y\in J$

substituce: $f(x): I\to J$ $\qquad = G(f(x)),\ x\in I$

(nemusí být prosté)

Př.: (1)
$$\int \frac{x\,dx}{x^8+1} = \frac{1}{2}\int \frac{2x\,dx}{(x^2)^4+1} \;\left|\begin{array}{l} y=x^2 \\ dy=2x\,dx \end{array}\right|$$

$$= \frac{1}{2}\int \frac{dy}{y^4+1}$$

(2)
$$\int \frac{dx}{\cos^3 x} = \int \frac{\cos x\,dx}{\underbrace{\cos^4 x}_{=(\cos^2 x)^2=(1-\sin^2 x)^2}} \;\left|\begin{array}{l} y=\sin x \\ dy=\cos x\,dx \end{array}\right| = \int \frac{dy}{(1-y^2)^2}$$

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2. VoS (Věta 5.4)

$$\int f(x)\,dx \;\left|\begin{array}{l} x=\varphi(t) \\ dx=\varphi'(t)\,dt \end{array}\right| = \int f(\varphi(t))\varphi'(t)\,dt = F(t)$$

$x\in I$ $\qquad$ $t\in J$

substituce: $\varphi(t): J\to I$ $\qquad = F(\varphi_{-1}(x))$

musí být prosté, $\varphi'(t)\in\mathbb{R}\setminus\{0\}$

aplikace: speciální substituce (viz níže)

Značení: $R = R(u,v)$ ... racionální fce prom. u, v

(tj.: pomocí $+, -, \cdot, :$, např.

$R = \frac{u+v}{u-v}$)

① $\int R\Big(x, \underbrace{\sqrt[r]{\frac{ax+b}{cx+d}}}_{t}\Big)\,dx$, $r \in \mathbb{N}$, $r \geq 2$

$a, b, c, d \in \mathbb{R}$

$(ad - bc \neq 0)$

substituce: $t = \sqrt[r]{\frac{ax+b}{cx+d}}$ $\qquad x = \frac{dt^r - b}{a - ct^r} = \varphi(t)$

$t^r = \frac{ax+b}{cx+d} \rightsquigarrow dx = \varphi'(t)\,dt$

$$\Longrightarrow \int R(\varphi(t), t)\,\varphi'(t)\,dt \text{ --- racionální funkce!!}$$

Př. $\int \frac{dx}{x+2\underbrace{\sqrt{x-1}}_{t}} = \left| \begin{array}{l} \sqrt{x-1} = t \\ x = t^2+1 \\ dx = 2t\,dt \end{array} \right| = \int \frac{2t\,dt}{t^2+1+2t} = \int \frac{2t\,dt}{(t+1)^2}$

$x \in (1, +\infty) \longleftrightarrow t \in (0, +\infty)$ $\qquad t \in (0, +\infty)$

$\varphi(t) = t^2+1$

Věta F: $\frac{2t}{(t+1)^2} = \frac{A}{t+1} + \frac{B}{(t+1)^2}$

$\Longleftrightarrow \quad 2t = A(t+1) + B$

$$\begin{array}{ll} t^0: & 0 = A + B \\ t^1: & 2 = A \end{array} \Longrightarrow A = 2,\ B = -2$$

$$\int \frac{2t\,dt}{(t+1)^2} = 2\int\frac{dt}{t+1} - 2\int\frac{dt}{(t+1)^2} =$$

$$= 2\cdot\ln|t+1| + \frac{2}{t+1}, \quad t\in(0,+\infty)$$

$$\Rightarrow \int\frac{dx}{x+2\sqrt{x-1}} = 2\cdot\ln\left(\sqrt{x-1}+1\right) + \frac{2}{\sqrt{x-1}+1}, \quad x\in(1,+\infty)$$

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② $\int R(x, \underbrace{\sqrt{ax^2+bx+c}}_{r(x)})\,dx$

α) $r(x)$ má reálné kořeny: λ, μ

$\rightarrow$ $r(x) = a(x-\lambda)(x-\mu) = \frac{a(x-\lambda)}{x-\mu}\cdot(x-\mu)^2$

$$\sqrt{r(x)} = \sqrt{\frac{a(x-\lambda)}{x-\mu}\cdot(x-\mu)^2} = \underbrace{\sqrt{\frac{a(x-\lambda)}{x-\mu}}}_{= t}\cdot|x-\mu|$$

t ... typ ①

Př: $\int \frac{dx}{(x+1)^5\underbrace{\sqrt{x^2+2x}}_{r(x)}}$ $\quad x\in\underset{+}{(0,+\infty)}, \underset{-}{(-\infty,-2)}$

rozklad: $x^2+2x = x(x+2) = \frac{x+2}{x}\cdot x^2$

$$\sqrt{x^2+2x} = \sqrt{\frac{x+2}{x}}\cdot\sqrt{x^2} = \pm x\underbrace{\sqrt{\frac{x+2}{x}}}_{= t}$$

substituce: $t=\sqrt{\frac{x+2}{x}}$

$x = \underbrace{\frac{2}{t^2-1}}_{=\varphi(t)}$, $dx = \frac{-4t}{(t^2-1)^2}\,dt$ $\Rightarrow$ $x+1 = \frac{t^2+1}{t^2-1}$

$$= \int \frac{(t^2-1)^5}{(t^2+1)^2} \cdot \frac{1}{\frac{2}{t^2-1}\cdot t} \cdot \frac{-4t}{(t^2-1)^2}\,dt = \int R(t)\,dt \ \ldots$$

B) $a > 0$: Eulerova substituce:

$$\sqrt{ax^2+bx+c} = t - \sqrt{a}\cdot x \quad \Big/^2$$

$$\cancel{ax^2} + bx + c = t^2 - 2\sqrt{a}\,t\cdot x + \cancel{ax^2}$$

$$\Rightarrow \quad bx + 2\sqrt{a}\,t\cdot x = t^2 - c$$

$$x = \frac{t^2-c}{b+2\sqrt{a}\,t} = \varphi(t)$$

$$dx = \varphi'(t)\,dt \ \ldots$$ racionální fce

Př. $\displaystyle\int \frac{dx}{(x+1)^5\sqrt{x^2+2x}}$

$$\sqrt{x^2+2x} = t - x$$

$$x^2 + 2x = t^2 - 2tx + x^2$$

$$\Rightarrow \quad x = \frac{t^2}{2(t+1)}, \quad dx = \frac{t(t+2)}{2(t+1)^2}\,dt$$

$$\sqrt{x^2+2x} = t - x = t - \frac{t^2}{2(t+1)} = \frac{t(t+2)}{2(t+1)}$$

$$= \int \frac{1}{\left(\frac{t^2}{2(t+1)}+1\right)^5 \cdot \frac{t(t+1)}{2(t+1)}} \cdot \frac{t(t+2)}{2(t+1)^2}\,dt = \int R(t)\,dt$$