Def. Nechť $\exists \delta > 0$ t.ž. $f(x)$ je definována na $\mathcal{U}(x_0, \delta)$. Derivací $f(x)$ rozumíme
$$\lim_{h \to 0} \frac{1}{h}\left(f(x_0+h) - f(x_0)\right).$$
Značíme: $f'(x_0)$, $\frac{df}{dx}(x_0)$, $f'(x)\Big|_{x=x_0}$.

Terminologie: $f'(x_0) \in \mathbb{R}$ ... vlastní der.

$f'(x_0) = \pm\infty$ ... nevlastní.

Pozn.:
- ekvivalentně: $\displaystyle\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ $(x = x_0 + h)$
- $f(x) = g(x)$ na jistém $\mathcal{U}(x_0)$ $\Rightarrow f'(x_0) = g'(x_0)$
- geometricky: směrnice tečny
- fyzikálně: rychlost změny
- $x \mapsto f'(x)$ je opět funkce, obecně $\mathcal{D}_{f'} \subset \mathcal{D}_f$, $\mathcal{H}_{f'} \subset \mathbb{R}^*$

Def. Derivace zprava / zleva: analogicky ($h \to 0\pm$ resp. $x \to x_0\pm$). Značíme $f'_\pm(x_0)$, $f'(x)\Big|_{x = x_0\pm}$

Příklady

⓪ $c' = 0$, neboť: $f(x) = c$, $\forall x \in \mathbb{R}$

$$\frac{1}{h}\left(f(x+h) - f(x)\right) = \frac{1}{h}\cdot(c-c) = 0, \quad \forall h \in P(0)$$

$(a \text{ tedy} \to 0,\ h \to 0)$

① $(x^n)' = nx^{n-1}$, $\forall x \in \mathbb{R}$, $n \in \mathbb{N}$

binomické věta: $(x+h)^n = \sum\limits_{k=0}^{n} \underbrace{\binom{n}{k} x^{n-k}}_{R_k} h^k$

$$= x^n + nx^{n-1}h + \sum_{k=2}^{n} R_k h^k$$

$$\Rightarrow \frac{1}{h}\left((x+h)^n - x^n\right) = nx^{n-1} + \sum_{k=2}^{n} R_k \underbrace{h^{k-1}}_{\substack{\to 0 \\ \text{VoAL}}}$$

$$\to nx^{n-1}, \quad h \to 0$$

Speciálně: $x' = 1$, $(x^2)' = 2x$, $x \in \mathbb{R}$.

② $\left(\frac{1}{x^m}\right)' = \frac{-m}{x^{m+1}}$, $\forall x \neq 0$, $m \in \mathbb{N}$

$$\frac{1}{h}\left(\frac{1}{(x+h)^m}-\frac{1}{x^m}\right)=\frac{1}{h}\left(\frac{x^m-(x+h)^m}{x^m(x+h)^m}\right)$$

$$=\underbrace{\frac{-1}{x^m(x+h)^m}}_{\text{VoAL}\searrow\ \frac{-1}{x^{2m}}}\cdot\underbrace{\frac{(x+h)^m-x^m}{h}}_{1\ \searrow\ mx^{m-1}}\longrightarrow\frac{-m}{x^{m+1}}\qquad(x\neq 0)$$

3 $(\sin x)'=\cos x\,,\ (\cos x)'=-\sin x\,,\ \forall x\in\mathbb{R}$

platí: $\sin\alpha-\sin\beta=2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

$\cos\alpha-\cos\beta=-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

$$\Rightarrow\frac{\sin(x+h)-\sin x}{h}=-\frac{2}{h}\cdot\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}$$

$$=\underbrace{\cos\left(x+\frac{h}{2}\right)}_{\downarrow\ \cos x}\cdot\underbrace{\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}}_{\to 1}\longrightarrow\cos x\,,\ h\to 0$$

dle: spojitost $\cos y$ & $\frac{\sin y}{y} \to 1, y \to 0$

$$\frac{\cos(x+h)-\cos x}{h} = -\frac{2}{h}\sin\left(x+\frac{h}{2}\right)\cdot\sin\frac{h}{2}$$

$$= -\sin\left(x+\frac{h}{2}\right)\cdot\frac{\sin\frac{h}{2}}{\frac{h}{2}} \to -\sin x$$

(4) $(\ln x)' = \frac{1}{x}$, pro $\forall x \in (0,+\infty)$

$$\frac{\ln(x+h)-\ln x}{h} = \frac{1}{h}\ln\left(\frac{x+h}{x}\right)$$

$$= \underbrace{\frac{\ln\left(1+\frac{h}{x}\right)}{\frac{h}{x}}}_{\to 1}\cdot\frac{1}{x} \to \frac{1}{x}, \; h \to 0$$

(5) $(e^x)' = e^x$, $\forall x \in \mathbb{R}$

$$\frac{1}{h}\Big(\underbrace{e^{x+h}}_{=\,e^x\cdot e^h} - e^x\Big) = e^x \cdot \underbrace{\frac{e^h-1}{h}}_{\to 1} \to e^x$$

6 $$(\operatorname{sgn} x)' = \begin{cases} 0, & x \neq 0 \quad (\alpha) \\ +\infty, & x = 0 \quad (\beta) \end{cases}$$

ad α): $\operatorname{sgn} x = c$ na $P(x_0, \delta)$, $x_0 \neq 0$

ad β): platí $x \cdot \operatorname{sgn} x = \dfrac{x}{\operatorname{sgn} x} = |x|$

$\forall x \neq 0$

$$\Rightarrow \frac{\operatorname{sgn} h - \operatorname{sgn} 0}{h} = \frac{1}{|h|} \to +\infty,\ h \to 0$$

(Věta 2.8)

7 $|x|' = \operatorname{sgn} x$, $x \neq 0$

$(|x|)'_{\pm}(0) = \pm 1$, a tedy $(|x|')(0)\ \nexists$

dk. $\dfrac{|h| - |0|}{h} = \operatorname{sgn} h \to \pm 1$, $h \to 0\pm$

(8) $(\sqrt{x})' = \frac{1}{2\sqrt{x}}, \; x > 0$

$(\sqrt{x})'_+(0) = +\infty$ (spojitost $\sqrt{x}$)

dk. $\frac{\sqrt{x}-\sqrt{x_0}}{x-x_0} = \frac{1}{\sqrt{x}+\sqrt{x_0}} \to \frac{1}{2\sqrt{x_0}}, \; x \to x_0$

$\frac{\sqrt{h}-\sqrt{0}}{h} = \frac{1}{\sqrt{h}} \to +\infty, \; h \to 0+$ (V. 2.8)

!! POZOR: $f(x)$ spojitá $\cancel{\Leftrightarrow}$ $\exists f'(x)$

Věta 4.1. $\exists f'(x_0) \in \mathbb{R} \Rightarrow f(x)$ spojitá v x_0

dk. Věta 2.5 $\Rightarrow$ stačí dokázat, že
$f(x) \to f(x_0), \; x \to x_0$

TRIK: $f(x) = f(x_0) + f(x) - f(x_0)$

$$= f(x_0) + \underbrace{\frac{f(x)-f(x_0)}{x-x_0}}_{\to L \in \mathbb{R}} \cdot \underbrace{(x-x_0)}_{\to 0} \to f(x_0)$$

dle VoAL

Věta 4.2 Nechť $\exists f'(x_0), g'(x_0)$ vlastní.

Pak: (1) $(f+g)'(x_0) = f'(x_0) + g'(x_0)$

(2) $(f-g)'(x_0) = f'(x_0) - g'(x_0)$

(3) $(f\cdot g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$

(4) $\left(\frac{f}{g}\right)'(x_0) = \frac{1}{(g(x_0))^2}\left(f'(x_0)g(x_0) - f(x_0)g'(x_0)\right)$

Dk. (1) $(f+g)(x_0+h) - (f+g)(x_0)$

$= f(x_0+h) + g(x_0+h) - f(x_0) - g(x_0)$

$= f(x_0+h) - f(x_0) + g(x_0+h) - g(x_0)$

(3) TRIK: $\frac{1}{h}\Big(f(x_0+h)g(x_0+h) - f(x_0)g(x_0)$

$\pm g(x_0+h)f(x_0)\Big)$

$$= \frac{f(x_0+h)-f(x_0)}{h}\cdot \underbrace{g(x_0+h)}_{\to g(x_0),\ \text{dle V. 4.1}}$$

$$+ f(x_0)\cdot\frac{g(x_0+h)-g(x_0)}{h} \to \text{PS (3)}$$

ad (4): nejprve (4)': $\left(\frac{1}{g}\right)' = -\frac{g}{g^2}$

$$\frac{1}{g(t_0+h)} - \frac{1}{g(t_0)} = \frac{g(t_0) - g(t_0+h)}{g(t_0+h)\,g(t_0)}$$

$$\Rightarrow \frac{1}{h}\left(\frac{1}{g(t_0+h)} - \frac{1}{g(t_0)}\right)$$

$$= \underbrace{\frac{-1}{g(t_0)\cdot g(t_0+h)}}_{\to \frac{1}{(g(t_0))^2},\ \text{V.4.1}} \cdot \underbrace{\frac{g(t_0+h) - g(t_0)}{h}}_{\to g'(t_0)}$$

(4)' $\Rightarrow$ (4) pomocí (3), neboť:

$$\left(\frac{f}{g}\right)' = \left(f\cdot\frac{1}{g}\right)' = f'\cdot\frac{1}{g} + f\left(\frac{1}{g}\right)'$$

$$= \frac{f'}{g} - \frac{fg'}{g^2} = \frac{1}{g^2}\left(f'g - fg'\right)$$

<u>Příkl.</u> 1 $(e^x \cdot \sin x)' = (e^x)' \sin x + e^x (\sin x)' = e^x \sin x + e^x \cos x,\ x \in \mathbb{R}$

2 pro $\forall x \neq \frac{\pi}{2} + k\pi$ $(\Leftrightarrow \cos x \neq 0)$

$$(\operatorname{tg} x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cos x - \sin x \cdot (\cos x)'}{\cos^2 x}$$

$$= \frac{\cos^2 x - (-\sin^2 x)}{\cos^2 x} = \frac{1}{\cos^2 x}$$

POZOR: (3), (4) obecně neplatí, jen mají-li PS smysl (v $\mathbb{R}^*$).

$$f(x) = \begin{cases} 1, & x = 0 \\ \frac{1}{x}, & x \neq 0. \end{cases}$$

$\Rightarrow f'(0) = +\infty$, neboť:

$$\frac{f(h) - f(0)}{h} = \frac{\frac{1}{h} - 1}{h} = (1 - h) \cdot \frac{1}{h^2} \to 1 \cdot (+\infty)$$

$(f^2)'(0) \nexists$, neboť $\dfrac{f^2(h)-f^2(0)}{h}$

$= \dfrac{\frac{1}{h^2}-1}{h} = (1-h^2)\cdot\dfrac{1}{h^3} \to \pm\infty,\ h\to 0\pm$

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leč formálně:

$(f\cdot f)'(0) = f'(0)\cdot f(0) + f(0)f'(0) = +\infty$

<u>Lemma 4.1</u>. Nechť $\exists f'(x_0) \neq 0$.

Pak $\exists \delta>0$ t. ž. $f(x) \neq f(x_0)$, $x\in P(x_0,\delta)$.

<u>Dk.</u> polož $\varphi(x) = \dfrac{f(x)-f(x_0)}{x-x_0}$, $x\neq x_0$

máme: $\varphi(x) \to A \in \mathbb{R}^*$, $A\neq 0$, $x\to x_0$

L. 2.1 $\Rightarrow \exists \delta>0$ t. ž. $\varphi(x)\neq 0$, $x\in P(x_0,\delta)$

leč: $f(x) = f(x_0) + \underbrace{\varphi(x)\cdot(x-x_0)}_{\neq 0,\ x\in P(x_0,\delta)}$