<u>Věta 4.3</u> nechť $\exists$ vlastní $f'(t_0)$ a $g'(y_0)$, kde $y_0 = f(t_0)$. Pak $(g\circ f)'(t_0) = g'(f(t_0))\cdot f'(t_0)$.

<u>Dk.</u> cíl (*) $$\frac{g(f(t)) - g(f(t_0))}{t - t_0} \to g'(y_0)\cdot f'(t_0)$$

<u>1.</u> nechť $f'(t_0) \neq 0$ .... L. 4.1 $\Rightarrow \exists\, \delta > 0$ t. ž. $f(t) \neq f(t_0)$, $t \in P(t_0, \delta)$

LS(*): $$\underbrace{\frac{g(f(t)) - g(f(t_0))}{f(t) - f(t_0)}}_{\downarrow\; g'(y_0)} \cdot \underbrace{\frac{f(t) - f(t_0)}{t - t_0}}_{\downarrow\; f'(t_0)}$$

dle VoLSF (Věta 2.5, (b))

... vnitřní fce: $f(t) \to f(t_0) = y_0$

(spojitost $\Leftarrow$ V. 4.1)

... vnější fce: $\varphi(y) = \dfrac{g(y) - g(y_0)}{y - y_0} \to g'(y_0)$

a tedy VoAL $\Rightarrow$ (*) platí

2. $f'(x_0)=0 \Rightarrow PS(*)=0$, tedy

stačí ukázat: $\dfrac{g(f(x))-g(f(x_0))}{x-x_0} \to 0,\ x\to x_0.$

$\varphi'(y) \to g'(y_0)\in\mathbb{R}$ ... L. 2.1 $\Rightarrow \exists K, \eta>0$

$y\to y_0 \qquad |\varphi(y)|\leq K,\ y\in P(y_0,\eta)$

$\Rightarrow |g(y)-g(y_0)|\leq K|y-y_0|$

... platí i pro $y\in U(y_0,\eta)$

$f(x)$ spojité v $x_0 \Rightarrow \exists\delta>0$ t.ž.

$f(x)\in U(f(x_0),\eta),\ \forall x\in U(x_0,\delta)$

---

celkem tedy: $x\in P(x_0,\delta) \Rightarrow$

$$\left|\frac{g(f(x))-g(f(x_0))}{x-x_0}\right| \leq \frac{K|f(x)-f(x_0)|}{|x-x_0|}$$

$$= K\left|\frac{f(x)-f(x_0)}{x-x_0}\right| \to 0,\ x\to x_0.$$

Příkl. 1) $(\cos(x^2))' = -\sin(x^2)\cdot 2x$, $x\in\mathbb{R}$

$\dots$ $g(y)=\cos y$, $f(x)=x^2$

2) $(\sqrt{x^2+1})' = \frac{1}{2\sqrt{x^2+1}}\cdot 2x = \frac{x}{\sqrt{x^2+1}}$, $x\in\mathbb{R}$

$\dots$ $g(y)=\sqrt{y}$, tj. $g'(y)=\frac{1}{2\sqrt{y}}$, $y>0$

$\dots$ $f(x)=x^2+1>0$

3) $(F(ax+b))' = F'(ax+b)\cdot a$

$\forall x$ t. ž. $\exists F'(y)$ v bodě $y=ax+b$

4) $(x^x)' = ?$ $\quad x^x = e^{x\cdot\ln x}$, tj.

$$(x^x)' = e^{x\cdot\ln x}\cdot\underbrace{(x\cdot\ln x)'}_{= 1\cdot\ln x + x\cdot\frac{1}{x}}$$

$$= x^x(\ln x+1), \quad \forall x>0$$

5) $|f(x)|' = f'(x)\cdot\operatorname{sgn} f(x)$, $f(x)\neq 0$

Věta 4.4 (Derivace inverzní fce.)

Dk. počítejme $\displaystyle\lim_{y \to y_0} \frac{f_{-1}(y) - f_{-1}(y_0)}{y - y_0}$

TRIK: $y = f(f_{-1}(y))$, $y \in \mathcal{U}(y_0, \delta) \subset J$

vnitřní bod!!

$$\Rightarrow Q(y) = \frac{f_{-1}(y) - f_{-1}(y)}{f(f_{-1}(y)) - f(f_{-1}(y_0))} = \frac{1}{\varphi(f_{-1}(y))}$$

kde $\displaystyle\varphi(t) = \frac{f(t) - f(t_0)}{t - t_0} \to f'(t_0)$,

pro $x \to t_0$

vnitřní fce: $f_{-1}(y) \to f_{-1}(y_0) = t_0$

leč $f_{-1}(y) \neq f_{-1}(y_0)$

$y \in P(y_0, \delta)$

(prostá fce!!)

(1), (2) $\Longleftarrow$ VOLSF & VoAL

ad (3): $\varphi(x) \to 0 = f'(x_0)$, leč

$\varphi(x) \underset{<}{>} 0$ pokud $f(x)$ rostoucí
klesající

… užijeme navíc Větu 2.8.

<u>Příkl.</u> (1) $(\sqrt[n]{y})' = ?$

a) <u>n sudé</u>: $f(x) = x^n$, $I = J = [0, +\infty)$

$$f'(x) = n x^{n-1}$$

$y \in J$ vnitřní, tj. $y \in (0, +\infty)$:

$$(\sqrt[n]{y})' = \frac{1}{n(\sqrt[n]{y})^{n-1}} = \frac{1}{n\sqrt[n]{y^{n-1}}} \qquad (*)$$

b) <u>n liché</u> podobně: $I = J = \mathbb{R}$

- $y \neq 0$ … vzorec $(*)$ platí
- $y = 0$ … $x = 0$, tj. $f'(x) = 0$, <u>rostoucí</u>

$$\Rightarrow (\sqrt[n]{y})'\Big|_{y=0} = +\infty \quad (\text{V. } 4.4, \text{ bod } (3))$$

② $(\arcsin y)' = \dfrac{1}{\sqrt{1-y^2}}$, $y \in (-1,1)$

$\cdots f(t) = \sin x$, $I = [-\frac{\pi}{2}, \frac{\pi}{2}]$, $J = [-1,1]$

$y \in J$ vnitřní; tj. $y \neq \pm 1 \Rightarrow$

$(\arcsin y)' = \dfrac{1}{\cos(\arcsin y)}$, neb $f'(t) = \cos t \neq 0$

$t \in (-\frac{\pi}{2}, \frac{\pi}{2})$

----

pomocný výpočet: $\sin^2 x + \cos^2 x = 1$

$\Rightarrow |\cos x| = \sqrt{\cos^2 x} = \sqrt{1-\sin^2 x}$, a tedy

$\cos(\arcsin y) = |\cos(\arcsin y)|$

$= \sqrt{1-(\sin(\arcsin y))^2} = \sqrt{1-y^2}$, $y \in (-1,1)$

---

③ $(\operatorname{arctg} y)' = \dfrac{1}{1+y^2}$, $y \in \mathbb{R}$

$\cdots f(x) = \operatorname{tg} x$, $I = (-\frac{\pi}{2}, \frac{\pi}{2})$, $J = \mathbb{R}$

$\Rightarrow (\operatorname{arctg} y)' = \dfrac{1}{\operatorname{tg}'(\operatorname{arctg} y)} = \cos^2(\operatorname{arctg} y)$

lec: $\sin^2 x + \cos^2 x = 1 \quad / : \cos^2 x \neq 0$

$t \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\operatorname{tg}^2 x + 1 = \dfrac{1}{\cos^2 x}$