<u>Věta 14.4</u> [O střední hodnotě] nechť $\Omega \subset \mathbb{R}^N$ je otevřená, konvexní, nechť $f(\underset{\sim}{x}) \in C^1(\Omega)$. Pak pro $\forall \underset{\sim}{a}, \underset{\sim}{b} \in \Omega \; \exists \underset{\sim}{c} \in (\underset{\sim}{a}, \underset{\sim}{b})$ t.ž.

$$f(\underset{\sim}{b}) - f(\underset{\sim}{a}) = \nabla f(\underset{\sim}{c})(\underset{\sim}{b} - \underset{\sim}{a})$$
$$= \sum_{j=1}^{N} \frac{\partial f}{\partial x_j}(\underset{\sim}{c})(b_j - a_j).$$

<u>Dk.</u> pomocné funkce:

$$\underset{\sim}{\varphi}(t) = \underset{\sim}{a} + t(\underset{\sim}{b} - \underset{\sim}{a}), \quad t \in [0,1]$$
$$g(t) = f(\underset{\sim}{\varphi}(t))$$

vidíme: $g(t)$ spojité v $[0,1]$

.... složení spojitých funkcí

$\underset{\sim}{\varphi}(t) : [0,1] \to \Omega$, $f(\underset{\sim}{x}) : \Omega \to \mathbb{R}$

navíc: $g(t)$ diferencovatelné $\forall t \in (0,1)$

... díky Větě 14.3, bod <u>2</u>:

$\underset{\sim}{\varphi}(t)$ zřejmě diferencovatelné $\forall t$

$f(\underset{\sim}{x})$ též díky Větě 14.2., $\forall \underset{\sim}{x} \in \Omega$

$$dg(t) = df(\underset{\sim}{\varphi}(t))\, d\underset{\sim}{\varphi}(t)$$

$$g'(t) = \nabla f(\underset{\sim}{\varphi}(t))\, \underset{\sim}{\varphi}'(t) = \nabla f(\underset{\sim}{\varphi}(t))(\underset{\sim}{b}-\underset{\sim}{a})$$

$$= \sum_{j=1}^{N} \frac{\partial f}{\partial x_j}(\underset{\sim}{\varphi}(t))\underbrace{\frac{\partial \varphi_j}{\partial t}(t)}_{b_j - a_j}$$

Věta 6.5 (Lagrange) $\Rightarrow$ $\exists\, \tau \in (0,1)$ t. ž.

$$g(1) - g(0) = g'(\tau)\cdot(1-0)$$

neboli:

$$f(\underbrace{\underset{\sim}{\varphi}(1)}_{\underset{\sim}{b}}) - f(\underbrace{\underset{\sim}{\varphi}(0)}_{\underset{\sim}{a}}) = \nabla f(\underbrace{\underset{\sim}{\varphi}(\tau)}_{\underset{\sim}{c}})(\underset{\sim}{b}-\underset{\sim}{a})$$

Věta 14.5. Nechť $f: U(\underset{\sim}{a}) \to \mathbb{R}$ je C^2, kde $\underset{\sim}{a} \in \mathbb{R}^2$. Potom $\frac{\partial^2 f}{\partial x_1 \partial x_2}(\underset{\sim}{a}) = \frac{\partial^2 f}{\partial x_2 \partial x_1}(\underset{\sim}{a})$

Dk. BÚNO $\underset{\sim}{a} = \underset{\sim}{0}$, $f = f(x,y)$

tj. chci dokázat: $\frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial^2 f}{\partial y \partial x}(0,0)$.

pomocné funkce (pro $h, k > 0$)

$$Q(h,k) = \frac{1}{hk}\Big[f(h,k) - f(h,0) - f(0,k) + f(0,0)\Big]$$

$$\varphi_k(h) = \frac{1}{k}\Big[f(h,k) - f(h,0)\Big]$$

$$\psi_h(k) = \frac{1}{h}\Big[f(h,k) - f(0,k)\Big]$$

… přepíšeme $Q(h,k)$ dvěma způsoby:

1. $$Q(h,k) = \frac{1}{h}\left[\frac{f(h,k)-f(h,0)}{k} - \frac{f(0,k)-f(0,0)}{k}\right]$$

$$= \frac{1}{h}\Big[\varphi_k(h) - \varphi_k(0)\Big]$$

(Lagrange) $= \varphi_k'(a)\,, \quad a \in (0,h)$

$$= \frac{1}{k}\left[\frac{\partial f}{\partial x}(a,k) - \frac{\partial f}{\partial x}(a,0)\right]$$

$\left(\text{neboť } \varphi_k'(h) = \frac{1}{k}\left[\frac{\partial f}{\partial x}(h,k) - \frac{\partial f}{\partial x}(h,0)\right]\right)$

… Lagrange ještě jednou (od 0 do k)

$$= \frac{\partial^2 f}{\partial y \partial x}(a,b)\,, \quad b \in (0,k)$$

neboť $\varphi_r'(a) = \left[\frac{1}{r}\frac{\partial f}{\partial x}(a,y)\right]_{y=0}^{y=r}$

$$= \underbrace{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)}_{\text{"}\frac{\partial^2 f}{\partial y \partial x}}(a,b)\,.$$

---

2. analogickou úvahou (opačné pořadí proměnných)

$$Q(h,r) = \frac{1}{r}\left[\psi_h(r) - \psi_h(0)\right]$$

$$\text{(Lagr.)} \quad = \psi_h'(d) \quad , \; d \in (0,r)$$

$$= \frac{1}{h}\left[\frac{\partial f}{\partial y}(h,d) - \frac{\partial f}{\partial y}(0,d)\right]$$

$$\text{(Lagr.)} \quad = \frac{\partial^2 f}{\partial x \partial y}(c,d) \; ; \; c \in (0,h).$$

---

3. CELKEM tedy:

$$Q(h,r) = \frac{\partial^2 f}{\partial y \partial x}(a,b) = \frac{\partial^2 f}{\partial x \partial y}(c,d)$$

$(h,k) \to (0,0) \Rightarrow (a,b) \to (0,0)$

$(c,d) \to (0,0)$

$f \in C^2 \ldots \quad \frac{\partial^2 f}{\partial y \partial x}(a,b) \to \frac{\partial^2 f}{\partial y \partial x}(0,0)$

$\frac{\partial^2 f}{\partial x \partial y}(c,d) \to \frac{\partial^2 f}{\partial x \partial y}(0,0)$

<u>Věta 14.6</u> [Taylorův rozvoj v $\mathbb{R}^N$.]

Nechť $f(\underset{\sim}{x}) : U(\underset{\sim}{a},\delta) \to \mathbb{R}$ je C^3, $\underset{\sim}{a} \in \mathbb{R}^N$.

Pak pro $\forall \underset{\sim}{h} \in U(\underset{\sim}{a},\delta)\ \exists\ \underset{\sim}{\theta} \in (\underset{\sim}{0}, \underset{\sim}{h})$ t.ž.

$$f(\underset{\sim}{h}) = f(\underset{\sim}{a}) + \sum_{i=1}^{N} \frac{\partial f(\underset{\sim}{a})}{\partial x_i} h_i + \frac{1}{2} \sum_{i,j=1}^{N} \frac{\partial^2 f(\underset{\sim}{a})}{\partial x_i \partial x_j} h_i h_j$$

$$\text{kde } R_3(\underset{\sim}{h}) = \frac{1}{3!} \sum_{i,j,k=1}^{N} \frac{\partial^3 f(\underset{\sim}{\theta})}{\partial x_i \partial x_j \partial x_k} h_i h_j h_k .$$

<u>Dodatek</u>: $R_3(\underset{\sim}{h}) = o(\|\underset{\sim}{h}\|^2)$, $\underset{\sim}{h} \to \underset{\sim}{0}$.

Dk. nechť $\underset{\sim}{h} \in U(\underset{\sim}{a}, \delta)$ je dáno

... pomocná fce $g(t) = f(\underbrace{\underset{\sim}{a} + t\underset{\sim}{h}}_{\underset{\sim}{\varphi}(t)})$

Taylor v $\mathbb{R}$ (Věta 8.4)

$$\Rightarrow \boxed{g(1) = g(0) + g'(0) + \frac{1}{2} g''(0) + R_3}$$

kde $R_3 = \frac{1}{3!} g^{(3)}(\tau)$, $\tau \in (0,1)$

... stačí vyjádřit jednotlivé členy:

$g(1) = f(\underset{\sim}{a} + \underset{\sim}{h})$, $g(0) = f(\underset{\sim}{a})$

... řetízkové pravidlo (Věta 14.3, bod 2.)

$$g'(t) = \frac{\partial}{\partial t} f(\underset{\sim}{\varphi}(t)) = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(\underset{\sim}{\varphi}(t)) \underbrace{\frac{\partial \varphi_i}{\partial t}(t)}_{h_i}$$

$$g''(t) = \frac{\partial}{\partial t} \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(\underset{\sim}{\varphi}(t)) h_i$$

$$= \sum_{i=1}^{N} \underbrace{\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial x_i}(\underset{\sim}{\varphi}(t))\right)}_{\sum_{j=1}^{N} \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)(\underset{\sim}{\varphi}(t)) \frac{\partial \varphi_j}{\partial t}(t)} \underbrace{h_i}_{\text{konstanta}}$$

$$\sum_{j=1}^{N} \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)(\underset{\sim}{\varphi}(t)) \frac{\partial \varphi_j}{\partial t}(t)\, h_i$$

Celkem tedy: $g''(t) = \sum_{i,j=1}^{N} \frac{\partial^2 f}{\partial x_i \partial x_j}(\underset{\sim}{\varphi}(t))\, h_i h_j$

... analogicky se odvodí:

$$g^{(3)}(t) = \sum_{i,j,k=1}^{N} \frac{\partial^3 f(\underset{\sim}{\varphi}(t))}{\partial x_i \partial x_j \partial x_k} h_i h_j h_k$$

Dosazením $t=0$ resp. $t=\tau$ plyne závěr, kde $\underset{\sim}{\theta} = \underset{\sim}{\varphi}(\tau) = \underset{\sim}{a} + \tau \underset{\sim}{h}$.

<u>Věta 14.7.</u> Nechť f má v bodě $a \in R^N$ lokální extrém. Potom pro $\forall i = 1, \dots N$ platí: jestliže $\frac{\partial f}{\partial x_i}(\underset{\sim}{a})$ existuje, je rovno 0.

<u>Dk.</u> obecněji, buď $\underset{\sim}{v} \in R^N$, $\underset{\sim}{v} \neq \underset{\sim}{0}$

pomocná fce $g(t) = f(\underset{\sim}{a} + t\underset{\sim}{v})$,

t blízké $0 \in R$

zřejmě: $g(t)$ má v $t=0$ lok. extrém

... Věta 6.3 $\Rightarrow$ $g'(0) = 0$ (pokud $\exists$)

$$\underbrace{g'(0)}_{= \frac{\partial f}{\partial \underset{\sim}{v}}(\underset{\sim}{a})}$$

Věta 14.8 [Existence lok. extrémů]

Dk. z Věty 14.6. (Taylor) lze psát:

$$f(\underset{\sim}{a}+\underset{\sim}{h}) = f(\underset{\sim}{a}) + \nabla f(\underset{\sim}{a})\underset{\sim}{h} + \tfrac{1}{2}Q(\underset{\sim}{h}) + R_3(\underset{\sim}{h})$$

... přičemž: $\nabla f(\underset{\sim}{a}) = \underset{\sim}{0}$

$$|R_3(\underset{\sim}{h})| \leq \frac{1}{6}\sum_{i,j,k} \underbrace{\left|\frac{\partial^3 f(\theta)}{\partial x_i \partial x_j \partial x_k}\right|}_{\leq C} \underbrace{|h_i||h_j||h_k|}_{\leq \|\underset{\sim}{h}\|^3}$$

(díky spojitosti)

$$\Rightarrow |R_3(\underset{\sim}{h})| \leq C_3\|\underset{\sim}{h}\|^3, \quad \underset{\sim}{h} \text{ malé}$$

tj. $-C_3\|\underset{\sim}{h}\|^3 \leq R_3(\underset{\sim}{h}) \leq C_3\|\underset{\sim}{h}\|^3$

---

ad 1. víme: $\exists C_1 > 0$ t.ž. $Q(h) \geq C_1\|h\|^2$ pro $\forall h \in \mathbb{R}^N$

a tedy, dle předchozího:

$$f(a+h) \geq f(a) + \frac{C_1}{2}\|h\|^2 - C_3\|h\|^3$$

$$= f(a) + \underbrace{\|h\|^2}_{\substack{>0 \\ \text{pro } h \neq 0}} \underbrace{\left(\frac{c_1}{2} - c_3\|h\|\right)}_{\substack{>0 \\ \text{pro } \|h\| < \frac{c_1}{2c_3}}}$$

$\Rightarrow f(a+h) > f(a)$, pokud $0 < \|h\| < \frac{c_1}{2c_3}$

---

ad 2. víme, že $Q(h) \leq -c_2\|h\|^2, \ \forall h$

a tedy, dle výše uvedeného:

$$f(a+h) = f(a) + \underbrace{\tfrac{1}{2}Q(h)}_{\substack{\leq \\ -c_2\|h\|^2}} + \underbrace{R_3(h)}_{\substack{\leq \\ c_3\|h\|^3}}$$

$$\leq f(a) - \frac{c_2}{2}\|h\|^2 + c_3\|h\|^3$$

$$= f(a) + \underbrace{\|h\|^2}_{\substack{>0 \\ \text{pro } h \neq 0}} \underbrace{\left(-\frac{c_2}{2} + c_3\|h\|\right)}_{\substack{<0 \text{ pro} \\ \|h\| < c_2/2c_3}}$$

$\Rightarrow f(a+h) < f(a)$, pro $0 < \|h\| < \frac{c_2}{2c_3}$

3. (i) víme: $\exists\, \underset{\sim}{v} \in \mathbb{R}^N$ s.ž. $Q(v) > 0$

$$f(\underset{\sim}{a} + t\underset{\sim}{v}) = f(\underset{\sim}{a}) + \frac{1}{2} Q(t\underset{\sim}{v}) + R_3(t\underset{\sim}{v})$$

platí: $Q(t\underset{\sim}{v}) = t^2 Q(\underset{\sim}{v}) = c_1 t^2$

$$|R_3(t\underset{\sim}{v})| \leq C \|t\underset{\sim}{v}\|^3 \leq c_3 t^3$$

$$\Rightarrow f(\underset{\sim}{a} + t\underset{\sim}{v}) \geq f(\underset{\sim}{a}) + \underbrace{t^2 \left(\frac{c_1}{2} - c_3 t \right)}_{>0 \text{ pro } 0 < t < \frac{c_1}{2c_3}}$$

$\Rightarrow$ $\underset{\sim}{a}$ NENÍ lokální maximum

(ii) dále víme: $\exists\, \underset{\sim}{w} \in \mathbb{R}^N : Q(\underset{\sim}{w}) < 0$

platí: $Q(t\underset{\sim}{w}) = t^2 Q(\underset{\sim}{w}) = -c_2 t^2$

$$R_3(tw) \leq c_3 t^3$$

$$\Rightarrow f(\underset{\sim}{a} + t\underset{\sim}{w}) \geq f(\underset{\sim}{a}) + t^2 \underbrace{\left(-\frac{c_2}{2} + c_3 t \right)}_{<0 \text{ pro } 0 < t < \frac{c_2}{2c_3}}$$

$\Rightarrow$ $\underset{\sim}{a}$ NENÍ lokální minimum

Věta 14.10. [O existenci extrémů v $\mathbb{R}^N$.]

1. $f: M \to \mathbb{R}$ spojité, $M \subset \mathbb{R}^N$ omezené, a uzavřené $\Rightarrow$ $\exists$ globální extrémy f vůči M.

2. $f: \mathbb{R}^N \to \mathbb{R}$ spojité, $f(x) \to +\infty$ pro $\|x\| \to +\infty$ $\Rightarrow$ $\exists$ globální minimum f v $\mathbb{R}^N$

3. (analogicky pro maximum)

Dk. 1. ... Věta 13.10. $\Rightarrow$ M kompaktní

Věta 13.8. $\Rightarrow$ $\exists$ glob. extrémy

2: buď $\underset{\sim}{a} \in \mathbb{R}^N$ libovolné, např. $\underset{\sim}{a} = \underset{\sim}{0}$

buď $K > 0$ t.ž. $f(\underset{\sim}{0}) < K$.

víme: $f(x) \to +\infty$ pro $\|x\| \to +\infty$

tj. $\exists R > 0 : f(x) > K$ pro $\|x\| > R$.

polož $B = \{x \in \mathbb{R}^N ; \|x\| \le R\}$

<u>tvrdíme</u>: B je omezené (jasné),

navíc uzavřené

(důkaz uzavřenosti B: lze psát

$B = \varphi^{-1}(J)$, kde $J = (-\infty, R]$ je

uzavřená množina

$\varphi : \mathbb{R}^N \to \mathbb{R}$, $\varphi(x) = \|x\|$ je spojité,

a užij Větu 13.5)

<u>jsem hotov</u>: $\exists \underset{\sim}{x}_0 \in B$ bod minima

(f na B), viz bod <u>1.</u>

ověř: $\underset{\sim}{0} \in B$, tedy $f(\underset{\sim}{x}_0) \le f(\underset{\sim}{0})$

zároveň platí: $f(\underset{\sim}{x}) > K > f(\underset{\sim}{0})$,

pro $\forall \underset{\sim}{x} \in B^c$ $\Rightarrow$ x_0 je <u>globální</u> min.