Věta 14.12 [VoIF – 1 rovnice]

Dk. 1. $\frac{\partial F}{\partial y}(\underset{\sim}{a}, b) \neq 0$ ... BÚNO > 0

... díky spojitosti $F, \frac{\partial F}{\partial y}$ ... $\exists \Delta, \delta > 0$

t. ž. $\frac{\partial F}{\partial y} > 0$ na $\Omega = \mathcal{U}(\underset{\sim}{a}, \delta) \times \mathcal{U}(b, \Delta)$

navíc platí: $F(\underset{\sim}{x}, b-\Delta) < 0$

$F(\underset{\sim}{x}, b+\Delta) > 0$ pro

$\forall \underset{\sim}{x} \in \mathcal{U}(\underset{\sim}{a}, \delta)$

2. pomocné fce: $\varphi_{\underset{\sim}{x}} : y \longmapsto F(\underset{\sim}{x}, y)$

($\underset{\sim}{x} \in \mathcal{U}(\underset{\sim}{a}, \delta)$ pevné)

dle předchozího platí:

(i) $\varphi_{\underset{\sim}{x}}(b-\Delta) < 0$, $\varphi_{\underset{\sim}{x}}(b+\Delta) > 0$

(ii) $\varphi'_{\underset{\sim}{x}}(y) = \frac{\partial F}{\partial y}(\underset{\sim}{x}, y) > 0$, $y \in (b-\Delta, b+\Delta)$

$\Rightarrow \exists!\, y \in (b-\Delta, b+\Delta)$ t. ž. $F(\underset{\sim}{x}, y) = 0$

existence $\Leftarrow$ Věta 2.16 (Darboux)
jednoznačnost $\Leftarrow$ Věta 6.10 (rostoucí, a tedy prosté)

... označme $\boxed{y = Y(\underset{\sim}{t}).}$

3. $\underbrace{\text{výpočet } \frac{\partial Y}{\partial t_i}}$ ... buď $x \in U(a,\delta)$
$t \in \mathbb{R}$ malé

VoSH : $F(\underset{\sim}{A}) - F(\underset{\sim}{B}) = DF(\underset{\sim}{C})(\underset{\sim}{A} - \underset{\sim}{B})$
(V.14.4)
$$= \sum_j \frac{\partial F}{\partial x_j}(\underset{\sim}{C})(A_j - B_j)$$
kde $\underset{\sim}{C} \in (\underset{\sim}{A}, \underset{\sim}{B})$

$$0 = F(\underbrace{x + te^i, Y(x + te^i)}_{A_t}) - F(\underbrace{x, Y(x)}_{B})$$

$$= DF(C_t)(A_t - B) \quad , \quad C_t \in (A_t, B)$$

přičemž platí: $DF = \left(\frac{\partial F}{\partial x_1}, \dots \frac{\partial F}{\partial x_N}, \frac{\partial F}{\partial y}\right)$

$$A_t - B = (\underset{\sim}{te^i}, Y(x + te^i) - Y(x))$$
$\sim (0, \dots, t, 0, \dots)$

CELKEM $\Longrightarrow$

$$0 = t\,\frac{\partial F}{\partial x_i}(c_t) + \frac{\partial F}{\partial y}(c_t)\big(Y(x+te^i) - Y(x)\big)$$

$$\boxed{(*)\quad \frac{1}{t}\big(Y(x+te^i) - Y(x)\big) = -\frac{\frac{\partial F}{\partial x_i}(c_t)}{\frac{\partial F}{\partial y}(c_t)}}$$

PS omezené $\Big(\frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial y}$ spojité, navíc $\frac{\partial F}{\partial y} \geq \varepsilon > 0\Big)$

tedy LS omezené, neboli:

$$\Big|\frac{1}{t}\big(Y(x+te^i) - Y(x)\big)\Big| \leq K$$

$$|Y(x+te^i) - Y(x)| \leq K|t|$$

odtud plyne: $Y(x+te^i) \to Y(x),\ t \to 0$

tedy $A_t \to (x, Y(x)) = B$

a také $C_t \to$ —"—,

a konečně limitou v $(*)$ výše máme:

$$\boxed{\frac{\partial Y}{\partial t_i}(t) = -\frac{\frac{\partial F}{\partial t_i}(t, Y(t))}{\frac{\partial F}{\partial y}(t, Y(t))} \qquad (**)}$$

4. $Y(t)$ je tudíž C^1

… $Y(t)$ spojité $\Leftarrow$ Věta 14.2, neboť $\frac{\partial Y}{\partial t_i}$ jsou omezené dle $(**)$

… $\frac{\partial Y}{\partial t_i}(t)$ spojité $\Leftarrow$ dle $(**)$, neboť PS je nyní složení spojitých funkcí

Věta 14.2 $\Rightarrow \exists\, dY(t)$

5. další hladkost $Y(t)$

nechť $F \in C^2$, tedy PS $(**)$ je C^1

$\Rightarrow \frac{\partial Y}{\partial t_i} \in C^1$, neboli $Y \in C^2$,

(pro obecné C^k indukcí)

Věta 14.9. [Multiplikátory – 1 vazba.]

Dk.: dle předpokladu $\nabla g(\mathbf{a}) \neq 0$,

kde $\nabla = \left(\frac{\partial}{\partial x_1}, \dots \frac{\partial}{\partial x_N}\right)$

BÚNO: $\frac{\partial g}{\partial x_N}(\mathbf{a}) \neq 0$

píšeme: $\mathbf{x} = (\bar{\mathbf{x}}, x_N)$, $\mathbf{a} = (\bar{\mathbf{a}}, a_N)$

kde $\bar{x} = (x_1, \dots, x_{N-1})$

$\bar{a} = (a_1, \dots, a_{N-1})$

Věta 14.12. (najdi $g(\underbrace{\bar{x}}_{\in \mathbb{R}^{N-1}}, \underbrace{x_N}_{y}) = 0$)

$\Rightarrow \exists\, \varphi(\bar{\mathbf{x}}) : U(\bar{a}, \delta) \to U(a_N, \Delta)$

třídy C^1 s.ž. $g(\bar{\mathbf{x}}, x_N) = 0$

$\Leftrightarrow x_N = \varphi(\bar{\mathbf{x}})$,

pro $(\bar{\mathbf{x}}, x_N)$ blízké $\mathbf{a}$.

neboli $\Gamma \cap U(\mathbf{a}) = \text{graf}\ \varphi$

pomocné fce $h(\bar{x}) = f(\bar{x}, \varphi(\bar{x}))$

pro $\bar{x} \in U(\bar{a}, \delta)$

zřejmě: $\bar{x} = \bar{a}$ je lokální extrém h

... Věta 14.7. $\Rightarrow$ $\nabla_{\bar{x}} h(\bar{a}) = \underline{0}$,

neboli $\boxed{\dfrac{\partial h}{\partial x_i}(\bar{a}) = 0, \quad i = 1, \dots, N-1}$ (*)

píšeme po složkách:

$$h(\bar{x}) = h(x_1, \dots, x_{N-1})$$

$$= f(x_1, \dots, x_{N-1}, \varphi(x_1, \dots, x_{N-1}))$$

$$\Rightarrow \frac{\partial h}{\partial x_i} = \frac{\partial f}{\partial x_i} + \frac{\partial f}{\partial x_N} \cdot \frac{\partial \varphi}{\partial x_i}$$

... dosadíme-li $\bar{x} = \bar{a}$, tj. $\varphi(\bar{x}) = a_N$

... tedy $(\bar{a}, \varphi(\bar{a})) = a$

$$\Rightarrow \boxed{\frac{\partial h}{\partial x_i}(\bar{a}) = \frac{\partial f}{\partial x_i}(a) + \frac{\partial f}{\partial x_N}(a) \frac{\partial \varphi}{\partial x_i}(\bar{a})}$$ (**)

dále, z důkazu Věty 14.12. máme:

$$\boxed{\frac{\partial \varphi}{\partial x_i}(\bar{a}) = -\frac{\frac{\partial g}{\partial x_i}(a)}{\frac{\partial g}{\partial x_N}(a)}, \quad i=1,\dots} \qquad (***)$$

Položme konečně $\lambda = \dfrac{\frac{\partial f}{\partial x_N}(a)}{\frac{\partial g}{\partial x_N}(a)}$

Tvrdíme, že $\nabla f(a) = \lambda \nabla g(a)$, tj.:

$$\frac{\partial f}{\partial x_i}(a) = \lambda \frac{\partial g}{\partial x_i}(a), \quad i=1,\dots,N$$

$i=N$ ... ihned z definice λ

$i=1,\dots N-1$ ... kombinací $(*)$, $(**)$, $(***)$

$$0 \overset{(*)}{=} \frac{\partial h}{\partial x_i}(\bar{a}) \overset{(**)}{=} \frac{\partial f}{\partial x_i}(a) + \underbrace{\frac{\partial f}{\partial x_N}(a) \cdot \underbrace{\frac{\partial \varphi}{\partial x_i}(\bar{a})}_{\overset{(***)}{=} -\frac{\frac{\partial g}{\partial x_i}(a)}{\frac{\partial g}{\partial x_N}(a)}}}_{= -\lambda \frac{\partial g}{\partial x_i}(a)}$$