__Věta 20.1.__ [Gaussova v $\mathbb{R}^3$.] Buď $\Omega \subset \mathbb{R}^3$ „rozumná" oblast, $\underset{\sim}{F} : \Omega \to \mathbb{R}^3$ třídy C^1 na jistém okolí $\overline{\Omega} = \Omega \cup \partial\Omega$. Nechť $\partial\Omega$ je zobecněná plocha, $\underset{\sim}{n} : \partial\Omega \to \mathbb{R}^3$ vnější (vůči Ω) normála. Potom:

$$\int_{\Omega} (\operatorname{div} \underset{\sim}{F})\, dx\, dy\, dz = \int_{\partial\Omega} (\underset{\sim}{F} \cdot \underset{\sim}{n})\, dS .$$

__dk.__ podobně jako v $\mathbb{R}^2$ (Věta 19.6)

- invariance vůči otočení a posunutí
- princip skládanky (sjednocení, průnik)
- linearita vůči $\underset{\sim}{F}$

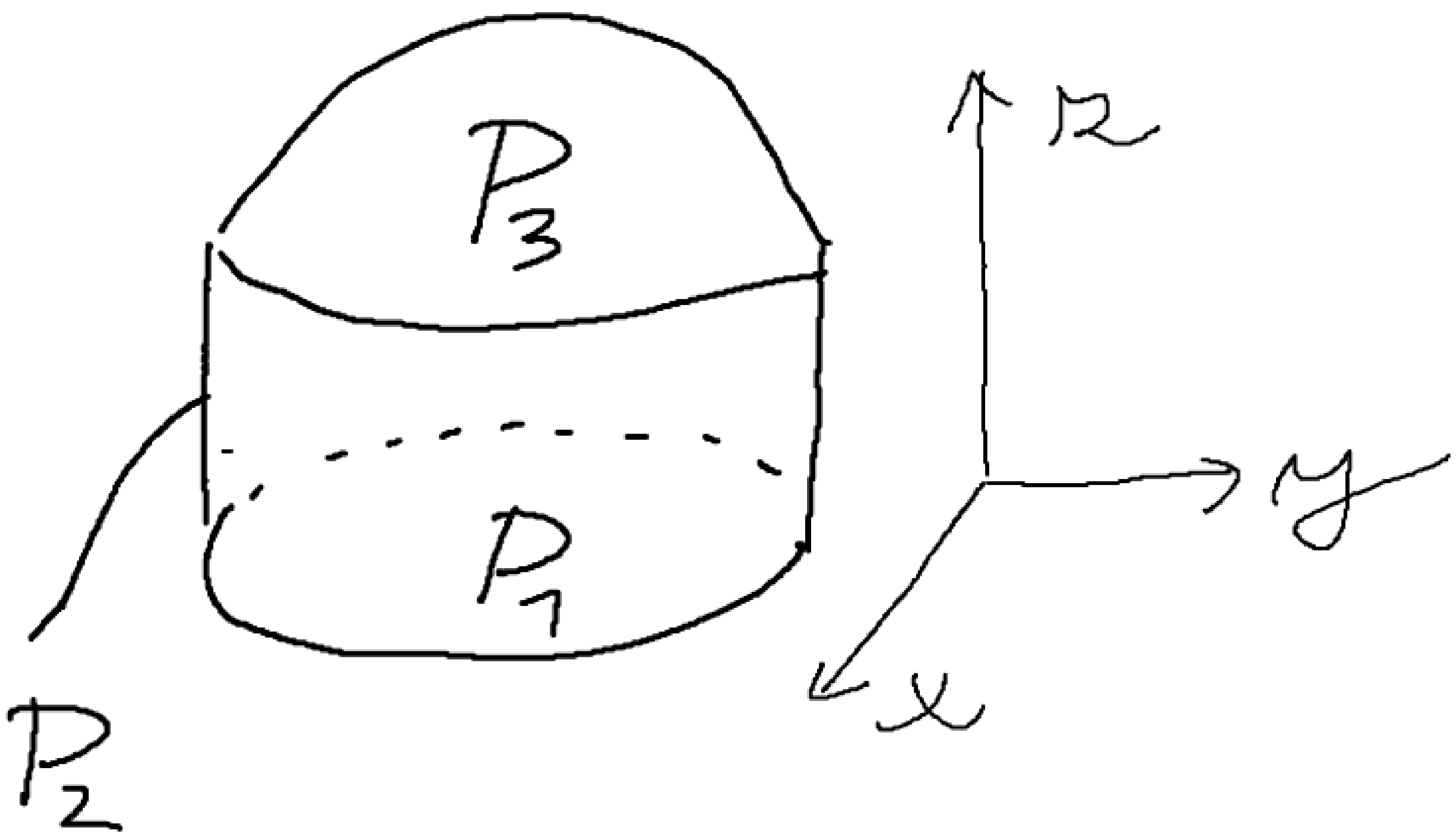
$\Longrightarrow$ BÚNO: $\underset{\sim}{F} = (0, 0, F_3)$

$$\Omega = \{ 0 < z < g(x,y) ;\ (x,y) \in \Pi \}$$

kde $\Pi \subset \mathbb{R}^2$ je omezená oblast,

$g : \Pi \to \mathbb{R}$ třídy C^1

cíl: $$\boxed{\int_\Omega \frac{\partial F_3}{\partial z}\, dx\, dy\, dz = \int_{\partial\Omega} F_3 n_3 \, dS}$$



<u>rozklad</u> $\partial\Omega = P_1 \cup P_2 \cup P_3$, kde

P_1 ... podstava (tj. fakticky Π),

$$\underset{\sim}{\varphi}(u,v) = (u,v,0), \quad (u,v)\in\Pi$$

$$\underset{\sim}{n} = (0,0,-1), \text{ tj. } n_3 = -1 \text{ na } P_1$$

$$\partial_u \underset{\sim}{\varphi} = (1,0,0), \quad \partial_v \underset{\sim}{\varphi} = (0,1,0)$$

$$\partial_u \underset{\sim}{\varphi} \times \partial_v \underset{\sim}{\varphi} = (0,0,1), \text{ a tedy}$$

$$dS = du\, dv$$

$$\Rightarrow \int_{P_1} F_3 n_3 \, dS = \int_{\Pi} F_3(u,v,0)\cdot(-1)\, du\, dv$$

$$= -\int_{\Pi} F_3(u,v,0)\, du\, dv$$

$P_2 = \{0 < z < g(x,y)\ ;\ (x,y) \in \partial\Omega\}$

pozorujme: $P_2 \perp$ rovině xy

tj. $\underline{n} \parallel$ s rovinou xy,

neboli $n_3 = 0$ na P_2

$$\Rightarrow \int_{P_2} F_3 n_3 \, dS = 0.$$

P_3 ... graf fce $g : \Pi \to \mathbb{R}$, tj.

$\underset{\sim}{\varphi}(u,v) = (u, v, g(u,v))\ ;\ (u,v) \in \Pi$

$\partial_u \underset{\sim}{\varphi} = (1, 0, \partial_u g)$

$\partial_v \underset{\sim}{\varphi} = (0, 1, \partial_v g)$

$$\partial_u \underset{\sim}{\varphi} \times \partial_v \underset{\sim}{\varphi} = \left(-\partial_u g, -\partial_v g, 1\right)$$

vnější normála

$$= \underset{\sim}{n} \circ \varphi \cdot \|\partial_u \underset{\sim}{\varphi} \times \partial_v \underset{\sim}{\varphi}\|$$

$$\Rightarrow \int_{P_3} F_3 n_3 \, dS$$

$$= \int_{\Omega} F_3 \circ \varphi \cdot \underbrace{n_3 \circ \varphi \cdot \|\partial_u \underset{\sim}{\varphi} \times \partial_v \underset{\sim}{\varphi}\|}_{1} \, du \, dv$$

$$= \int_{\Omega} F_3(u, v, g(u, v)) \, du \, dv.$$

CELKEM: $(u, v \to x, y)$

P.S. cíle: $\left(\int_{P_1} + \int_{P_3}\right) F_3 n_3 \, dS$

$$= \int_{\Omega} F_3(x, y, g(x, y)) - F_3(x, y, 0) \, dx \, dy$$

L.S. cíle: $\displaystyle\int_\Omega \frac{\partial F_3}{\partial z}\,dx\,dy\,dz = \left| \begin{array}{l}\text{Fubini} \\ (\text{V. } 18.3)\end{array}\right.$

$$= \int_\Pi \Big(\underbrace{\int_0^{g(x,y)} \frac{\partial F_3}{\partial z}(x,y,z)\,dz}_{} \Big)\,dx\,dy = \text{P.S.}$$

$$\Big[F_3(x,y,z) \Big]_{z=0}^{z=g(x,y)} = F_3(x,y,g(x,y)) - F_3(x,y,0)$$

<u>Věta 20.2.</u> [Vztah z. i. 1. a 2. druhu]

<u>dk.</u> BÚNO: P ... jednoduchá plocha

$\varphi(u,v)$ ... parametrizace ve shodě s orientací

$u,v \in \Pi$

$\Longrightarrow$

$$\partial_u \varphi \times \partial_v \varphi = \underset{\sim}{n} \circ \varphi \cdot \| \partial_u \varphi \times \partial_v \varphi \|$$

platí v celém Π

a tedy: $$P.S. = \int_P (\boldsymbol{F} \cdot \boldsymbol{n})\, dS$$

$$= \int_\Omega (\boldsymbol{F} \circ \boldsymbol{\varphi}) \cdot \underbrace{(\boldsymbol{n} \circ \boldsymbol{\varphi}) \cdot \| \partial_u \boldsymbol{\varphi} \times \partial_v \boldsymbol{\varphi} \|}_{\partial_u \boldsymbol{\varphi} \times \partial_v \boldsymbol{\varphi}}\, du\, dv$$

$$= \int_\Omega (\boldsymbol{F} \circ \boldsymbol{\varphi}) \cdot (\partial_u \boldsymbol{\varphi} \times \partial_v \boldsymbol{\varphi})\, du\, dv$$

$$= \int_P \boldsymbol{F} \cdot d\boldsymbol{S} = L.S.$$

Věta 20.3. [Grammův determinant.]

Nechť $P \subset \mathbb{R}^3$ je jednoduchá plocha, $\boldsymbol{\varphi} : \Omega \to P$ parametrizace, $f : P \to \mathbb{R}$.

Potom: $$\int_P f\, dS = \int_\Omega f \circ \boldsymbol{\varphi} \cdot \sqrt{g}\, du\, dv,$$

kde $$g = \det \begin{pmatrix} \partial_u \varphi \cdot \partial_u \varphi, & \partial_u \varphi \cdot \partial_v \varphi \\ \partial_u \varphi \cdot \partial_v \varphi, & \partial_v \varphi \cdot \partial_v \varphi \end{pmatrix}.$$

dk. plyne ihned ze vzorečku:

$$\boxed{\|\underset{\sim}{\alpha} \times \underset{\sim}{\beta}\|^2 = \det\begin{pmatrix} \underset{\sim}{\alpha}\cdot\underset{\sim}{\alpha}, & \underset{\sim}{\alpha}\cdot\underset{\sim}{\beta} \\ \underset{\sim}{\alpha}\cdot\underset{\sim}{\beta}, & \underset{\sim}{\beta}\cdot\underset{\sim}{\beta} \end{pmatrix}}$$

pro $\underset{\sim}{\alpha}, \underset{\sim}{\beta} \in R^3$

volíme $\underset{\sim}{\alpha} = \partial_u \underset{\sim}{\varphi}$, $\underset{\sim}{\beta} = \partial_v \underset{\sim}{\varphi}$

$\Rightarrow \|\partial_u \underset{\sim}{\varphi} \times \partial_v \underset{\sim}{\varphi}\| = \sqrt{g}$.

Lemma 20.1. [Reparametrizace plochy.]

dk.

$$\begin{tikzcd} & P & \\ \Pi \arrow[ur, "\varphi"] & & O \arrow[ul, "\psi"'] \arrow[ll, "\omega"] \end{tikzcd}$$

$x = (x_1, x_2, x_3)$

$u = (u_1, u_2)$ $\qquad$ $v = (v_1, v_2)$

1. KROK polož $\boxed{\omega = \varphi_{-1} \circ \psi}$ $\Longrightarrow$

$\omega : O \to \Pi$ spojité, 1-1, $\varphi \circ \omega = \psi$

2. KROK $\quad \omega \in C^1(O)$?

... stačí ověřit na okolí bodu v^0,
(kde $v^0 \in O$ je pevné, libovolné)

označme: $u^0 = \omega(v^0) \in \Pi$,

tj. $u^0 = \varphi_{-1}(x^0)$, $x^0 = \psi(v^0) \in P$

víme: $h(\nabla\varphi(u^0)) = 2$, tj.

$$\nabla\varphi = \left.\begin{pmatrix} \partial_1\varphi_1, \partial_1\varphi_2, \partial_1\varphi_3 \\ \partial_2\varphi_1, \partial_2\varphi_2, \partial_2\varphi_3 \end{pmatrix}^T\right|_{u=u^0}$$

má regulární subdeterminant řádu 2×2, BÚNO nechť to je

$$\left.\begin{pmatrix} \partial_1\varphi_1 & \partial_2\varphi_1 \\ \partial_1\varphi_2 & \partial_2\varphi_2 \end{pmatrix}\right|_{u=u^0} = \nabla\hat{\varphi}(u^0)$$

kde $\hat{\varphi} = \Pi\varphi$; $\Pi : \mathbb{R}^3 \to \mathbb{R}^2$

je projekce $(x_1, x_2, x_3) \longmapsto (x_1, x_2)$.

CELKEM: $\hat{\varphi}: \Omega \to \mathbb{R}^2$, třídy C^1

$D\hat{\varphi}(u^0)$ regulární

Věta 14.14. (o inverzní fci) $\Rightarrow$ $\exists$ $\mathcal{U}, \mathcal{W}$ otevřené

$u^0 \in \mathcal{U}$, $\hat{\varphi}(u^0) \in \mathcal{W}$

$\hat{\varphi}: \mathcal{U} \to \mathcal{W}$ je 1-1

a $\hat{\varphi}_{-1}$ je zde C^1

---

klíčový krok: volme V okolí v^0

malé t.ž. $\Pi\psi(V) \subset \mathcal{W}$

$\Rightarrow$ mohu psát: $\boxed{w = \hat{\varphi}_{-1} \circ (\Pi\psi)}$ všude na V

odsud ale ihned $w \in C^1(V)$

(složení C^1 funkcí)

3. KROK $\quad J\omega \neq 0$?

máme: $\quad \psi = \varphi \circ \omega$

$$D\psi = \big[(D\varphi)\circ\omega\big]\,\big[D\omega\big]$$

(Věta 14.3) $\qquad$ součin matic

?? $J\omega = 0$ ... pak $h(D\omega) \leq 1$

tedy $h(P.S.) \leq 1$

$\Rightarrow h(L.S.) = h(D\psi) \leq 1$

SPOR s vlastností (ii) parametrizace.

Dodatek: výše jsme ukázali:

$$\big(\partial_1\psi \,\big|\, \partial_2\psi\big) = \Big[\big(\partial_1\varphi \,\big|\, \partial_2\varphi\big)\circ\omega\Big]\,\big(D\omega\big)$$

$$\Rightarrow \quad \partial_1\psi \times \partial_2\psi = \big[(\partial_1\varphi \times \partial_2\varphi)\circ\omega\big]\cdot J\omega$$

tj: $\partial_1\psi \times \partial_2\psi$, $\partial_1\varphi \times \partial_2\varphi$ mají stejný/opačný směr $\Longleftrightarrow J\omega > 0 / < 0$.