Tvrzení. nechť $f \in L^1(a,b)$, kde $a < b \in \mathbb{R}$.
Pak $\forall \varepsilon > 0\ \exists \tilde{f}$ „pěkné", tj. $\tilde{f}, \tilde{f}'$ jsou spojité, omezené dokonce v $[a,b]$

a že: $\int_a^b |f(x) - \tilde{f}(x)|\, dx < \varepsilon$

Lemma 21.4. [Riemann–Lebesgue]

nechť $f \in L^1(a,b)$. Pak $\int_a^b f(x) \cos kx\, dx \to 0$,
$(\sin kx)$ pro $k \to \infty$.

Dk. 1. KROK: nechť f je „pěkné", tj. f, f' spojité,

navíc: $|f(x)|, |f'(x)| \le M \quad \forall x \in [a,b]$

$$\int_a^b f(x) \cdot \cos kx\, dx = \left[f(x) \cdot \frac{\sin kx}{k} \right]_{x=a}^{x=b} - \int_a^b f'(x) \frac{\sin kx}{k}\, dx$$

(per partes)

$$= P_1 + P_2$$

odhady: $|P_1| \le \left| f(b) \frac{\sin kb}{k} \right| + \left| f(a) \frac{\sin ka}{k} \right| \le \frac{2M}{k}$

$$|P_2| \le \int_a^b |f'(x)| \left| \frac{\sin kx}{k} \right| dx \le \frac{M(b-a)}{k}$$

tedy zřejmě: $\left| \int_a^b f(x) \cos kx\, dx \right| \le |P_1| + |P_2| \to 0$, $k \to \infty$

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2. KROK: $f \in L^1(a,b)$ obecné; chceme dokázat:

$$\forall \varepsilon > 0\ \exists k_0\ \forall k \ge k_0 : \left| \int_a^b f(x) \cdot \cos kx\, dx \right| < \varepsilon$$

nechť $\varepsilon > 0$ je dáno:

dle Tvrzení $\exists \tilde{f}$ pěkné s.ž. $\int_a^b |f(x)-\tilde{f}(x)|\,dx < \frac{\varepsilon}{2}$ (*)

a dle KROKU 1 $\int_a^b \tilde{f}(x)\cdot\cos kx\,dx \to 0$, pro $k\to\infty$

tedy $\exists k_0\ \forall k\geq k_0: \left|\int_a^b \tilde{f}(x)\cdot\cos kx\,dx\right| < \frac{\varepsilon}{2}$ (**)

CELKEM pro $k\geq k_0$ máme:

$$\int_a^b f(x)\cdot\cos kx = \int_a^b (f(x)-\tilde{f}(x))\cdot\cos kx + \int_a^b \tilde{f}(x)\cdot\cos kx$$

$$= I_1 + I_2,$$

odhady: $|I_1| \leq \int_a^b |f(x)-\tilde{f}(x)|\cdot\underbrace{|\cos kx|}_{\leq 1}\,dx$

$$\leq \int_a^b |f(x)-\tilde{f}(x)|\,dx < \frac{\varepsilon}{2} \text{ díky (*)}$$

$|I_2| < \frac{\varepsilon}{2}$ dle (**)

a tedy $\left|\int_a^b f(x)\cdot\cos kx\right| \leq |I_1| + |I_2| < \varepsilon$

Věta 21.4 [Parsevalova rovnost]

Nechť $f(x) \in L^2_{per}(0,2\pi)$. Pak $\displaystyle\frac{1}{\pi}\int_0^{2\pi} |f(x)|^2\,dx = \frac{a_0^2}{2} + \sum_{k=1}^{\infty}\left(a_k^2+b_k^2\right)$

Dk. (formální): $\displaystyle f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k\cos kx + b_k\sin kx\right)$

$$\Rightarrow \int_0^{2\pi} (f(x))^2\,dx =$$

$$= \int_0^{2\pi}\left(\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos kx + b_k\sin kx\right)\left(\frac{a_0}{2} + \sum_{l=1}^{\infty} a_l\cos lx + b_l\sin lx\right)dx$$

$$= \int_0^{2\pi}\left(\frac{a_0}{2}\right)^2 dx + \cancel{\frac{a_0}{2}\sum_{l=1}^{\infty}\int_0^{2\pi} a_l\cos lx + b_l\sin lx\,dx}$$

$$\cancel{+\frac{a_0}{2}\sum_{k=1}^{\infty}\int_0^{2\pi} a_k\cos kx + b_k\sin kx\,dx}$$

$$+ \sum_{k,l=1}^{\infty}\int_0^{2\pi} \underbrace{a_k a_l\cos kx\cdot\cos lx}_{\text{pro } k=l} + \cancel{a_k b_l\cos kx\cdot\sin lx}$$

$$\cancel{+ b_k a_l\sin kx\cdot\cos lx} + \underbrace{b_k b_l\sin kl\,\sin lx}_{\text{pro } k=l}\,dx$$

dle Lemmatu 21.1

zůstane pouze: $\displaystyle = \int_0^{2\pi}\left(\frac{a_0}{2}\right)^2 + a_k^2\cos^2 kx + b_k^2\sin^2 kx\,dx$

$$= \pi\frac{a_0^2}{2} + \sum_{k=1}^{\infty}\left(\pi a_k^2 + \pi b_k^2\right)$$