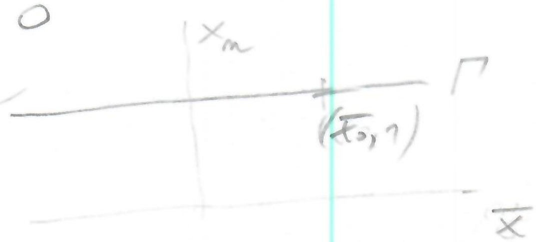
① $x\cdot\nabla u = \alpha$; $u=u(x)$, $x=(x_1,\dots,x_n)=(\bar{x},x_n)$

$u|_{x_n=1} = h$ ($x_n>0$)



$x' = x$, $x(0)=(y_0,1)$

$z' = \alpha$ $\quad z(0)=h(y)$

$\Rightarrow x(t) = (\bar{x}(t), x_n(t)) = (\bar{x}_0 e^t, e^t)$; Zi: $\frac{\bar{x}}{x_n} = \bar{C}$

$z(t) = h(\bar{x}_0) + \alpha \underset{\ln x_n}{t}$; $u(t) = U\left(\frac{\bar{x}}{x_n}\right)$

$x = (\bar{x}, x_n) \overset{?}{=} (\bar{x}_0 e^t, e^t)$; Zi: $x_n = e^t \iff t = \ln x_n$

$u(x) = z(t) = \boxed{h\left(\frac{\bar{x}}{x_n}\right) + \alpha \ln x_n}$ $\quad \bar{x}_0 = \bar{x}/x_n$

② $y\,\partial_x u + x\,\partial_y u = \gamma u$, $\gamma>0$, $(x,y)\in\mathbb{R}^2$

(i) $\gamma=0$ (homogenní rce)

$x' = y$ $\Rightarrow x'' = y' = x$; $\lambda = \pm 1$; F.S. $\{e^t, e^{-t}\}$

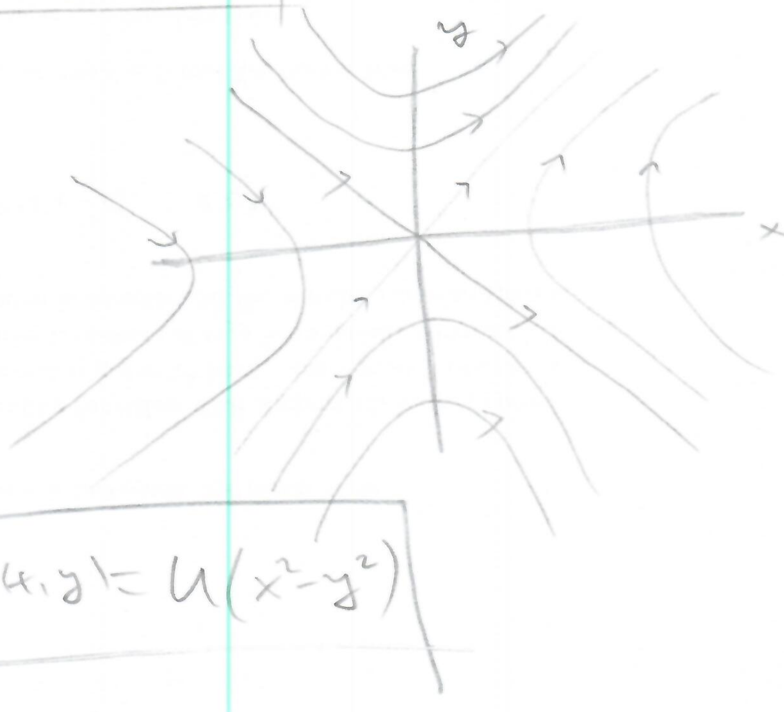
$y' = x$

$$\boxed{\begin{aligned} x(t) &= x_0\cosh t + y_0 \sinh t \\ y(t) &= x_0 \sinh t + y_0 \cosh t \end{aligned}}$$

nebo $\{\cosh t, \sinh t\}$

lépe: $y' = \frac{dy}{dt} = x$

$x' = \frac{dx}{dt} = y$



$\frac{dy}{dx} = \frac{x}{y}$

$y\,dy = x\,dy$

$\frac{y^2}{2} = \frac{x^2}{2} + c$

$\boxed{y^2 - x^2 = c}$

$\boxed{u(x,y) = U(x^2-y^2)}$

(iii) $y\partial_x u + x\partial_y u = \gamma u$

$u(x,0) = g(x)$

$$\boxed{\begin{array}{ll} x' = y & x(0) = x_0 \\ y' = x & y(0) = 0 \end{array}}$$

$z' = \gamma z \qquad z(0) = g(x_0) \qquad \Rightarrow \quad z(t) = g(x_0)e^{\gamma t}$

$$\left.\begin{array}{l} x'' - x = 0 \\ x(0) = x_0 \\ x'(0) = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x(t) = x_0 \cosh t = x \\ y(t) = x_0 \sinh t = y \end{array}$$

$x_0^2(\cosh^2 t - \sinh^2 t) = x^2 - y^2$

$|x_0| = \sqrt{x^2 - y^2}$

cos: $\operatorname{tgh} t = \frac{y}{x}$

$t = \operatorname{argtgh}\left(\frac{y}{x}\right) = \frac{1}{2}\ln\left(\frac{1+\frac{y}{x}}{1-\frac{y}{x}}\right) = \ln\sqrt{\frac{x+y}{x-y}}$

$$\Rightarrow \boxed{u(x,y) = g\left(\sqrt{x^2-y^2}\right)\cdot\left(\frac{x+y}{x-y}\right)^{\gamma/2}}$$

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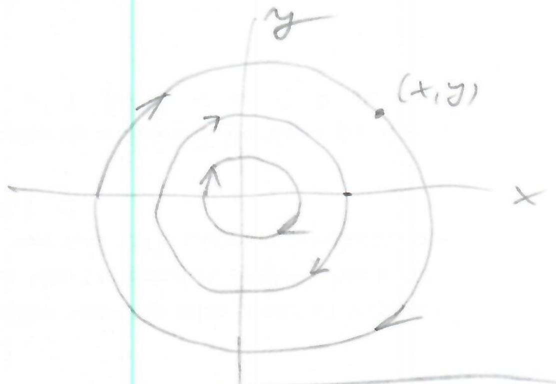
3 $y\partial_x u - x\partial_y u = \gamma u$



$\begin{array}{l} x' = y \\ y' = -x \end{array} \Rightarrow x^2 + y^2 = C$

$x'' = y' = -x; \quad x'' + x = 0$

$x(0) = x_0,$

$x'(0) = y_0$

$$\boxed{u(x,y) = U(x^2+y^2)}$$

(ii) $u(x_0,0) = g(x_0)$, $x_0 \in \mathbb{R}$

$x(t) = x_0 \cos t \qquad |x_0| = \sqrt{x^2+y^2}\ ; \quad t = \operatorname{arctg}\left(-\frac{y}{x}\right), x > 0$

$y(t) = -x_0 \sin t$

$z(t) = g(x_0)e^{\gamma t}$

$$\boxed{u(x,y) = g\left(\sqrt{x^2+y^2}\right)\exp\left(\gamma \operatorname{arctg}\left(-\frac{y}{x}\right)\right)}$$

(4) $x\partial_x u + y\partial_y u + (x^2+y^2)\partial_z u = 0 \quad , \quad (x,y,z)\in\mathbb{R}^3$

$$\Rightarrow \left.\begin{array}{l} x' = x \\ y' = y \\ z' = x^2+y^2 \end{array}\right\} \Rightarrow \begin{array}{l} x(t) = c_1 e^t \\ y(t) = c_2 e^t \\ z(t) = c_3 + \left(\frac{c_1^2}{2} + \frac{c_2^2}{2}\right) e^{2t} \end{array}$$

elimino $\underline{t}$: $\quad \frac{x}{y} = C \quad$ nelo $\quad (x^2+y^2) - 2z = C$

formálně: $\quad \underset{\sim}{a}(x) \perp \nabla u \quad ; \quad \underset{\sim}{a} = (x, y, x^2+y^2)$

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{y}{x} \qquad \frac{dz}{dx} = \frac{z'}{x'} = \frac{x^2+y^2}{x} = x + \underbrace{\frac{y^2}{x^2}}_{= C}\cdot x$$

$$\frac{dy}{y} = \frac{dx}{x} \Rightarrow \ln|y| = \ln|x| + C$$

? $\qquad dz = (C+1)\,x$

$$z = (C+1)\frac{x^2}{2}.$$

---

(5) $\partial_x u + xz\,\partial_y u - xy\,\partial_z u = 0$

(CH) $\quad x' = 1 \qquad x = t + c_1$

$y' = xz \qquad \frac{dz}{dy} = \frac{z'}{y'} = \frac{-xy}{xz} = -\frac{y}{z}$

$z' = -xy$

$$\Rightarrow z^2 + y^2 = C_2^2$$

$$\frac{dy}{dx} = \frac{y'}{x'} = xz = x\sqrt{C_2 - y^2} \qquad ; \ z > 0$$

$$\int \frac{dy}{\sqrt{C_2^2 - y^2}} = \int x\,dx$$

$$\cos(\arcsin\alpha) = \sqrt{1-\sin^2(\arcsin\alpha)} = \sqrt{1-\alpha^2}$$

$$\arcsin\left(\frac{y}{C_2}\right) = \frac{x^2}{2} + C_3$$

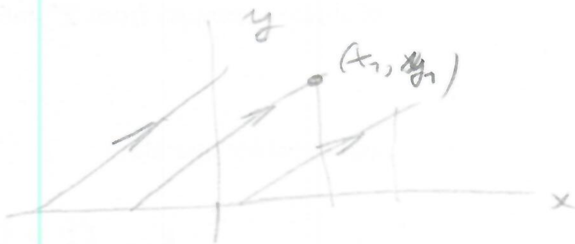
$$\arcsin\left(\frac{y}{\sqrt{z^2+y^2}}\right) - \frac{x^2}{2} = C_3 \quad \Big|\ \sin$$

$$\frac{y}{\sqrt{z^2+y^2}}\cos\left(\frac{x^2}{2}\right) - \cos\left(\arcsin\left(\frac{y}{\sqrt{z^2+y^2}}\right)\right)\cdot\sin\left(\frac{x^2}{2}\right) = C$$

(6) [Evans, ex. 2, p. 101] $\partial_x u + \partial_y u = u^2$ in $\{y>0\}$

$u = g$ on $\{y=0\}$

$$\left.\begin{array}{l} x' = 1,\ y' = 1 \\ z' = z^2 \\ x(0) = x_0 \\ y(0) = 0 \\ z(0) = z_0 = g(x_0) \end{array}\right\} \Rightarrow \begin{array}{l} x = c + t \\ y = t \\ z = \dfrac{z_0}{1 - t z_0} \end{array}$$



~~x_1, y_1 ... dehno.~~

$z_0 = g(x_0)$

$t = y_0$, $c = x_0 - y_0$

$z_0 = g(x_1 - y_1)$

$$\Rightarrow u(x_0, y_0) = \frac{g(x_0 - y_0)}{1 - y_0\, g(x_0 - y_0)}; \quad y_0\, g(x_0 - y_0) < 1$$