**Věta III.1.** [Energetická rovnost pro (VR).]

**Dk.** cíl: $e'(t) = 0$, pro $\forall t \in [0,T]$,

kde $e(t) := \displaystyle\int_\Omega (\partial_t u)^2(x,t) + |\nabla u|^2(x,t)\,dx$.

$$\Rightarrow \quad e'(t) = \frac{d}{dt}\int_\Omega (\partial_t u)^2 + |\nabla u|^2\,dx$$

$$= \int_\Omega \frac{\partial}{\partial t}\left\{(\partial_t u)^2 + |\nabla u|^2\right\}dx$$

$$= \int_\Omega 2\,\partial_t u\,\partial_{tt} u + \underbrace{2\nabla u \cdot \nabla(\partial_t u)}_{(*)}\,dx$$

$$(*) \overset{\uparrow}{=} \underbrace{\int_{\partial\Omega} 2\frac{\partial u}{\partial n}\,\partial_t u\,d\sigma}_{0} - \int_\Omega 2\Delta u\,\partial_t u\,dx$$

Greenova identita

neboť: $\frac{\partial u}{\partial n} \equiv 0$ na $\partial\Omega$, nebo

$u \equiv 0$ a tedy

$\partial_t u \equiv 0$ na $\partial\Omega$

tedy lze pokračovat:

$$e'(t) = \int_\Omega 2\,\partial_t u\, \partial_{tt} u - 2\Delta u\, \partial_t u \, dx$$

$$= 2\int_\Omega \underbrace{(\partial_{tt} u - \Delta u)}_{0} \partial_t u \, dx = 0$$

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**Věta III.2.** [O šíření vlny.]

**Dk.** cíl: $e'(t) \leq 0$, $t \in [0,R]$.

kde $e(t) = \displaystyle\int_{B(x_0,R-t)} (\partial_t u)^2(x,t) + |\nabla u|^2(x,t)\, dx$

$$\Rightarrow \quad e'(t) = \frac{d}{dt}\int_{B(x_0,R-t)} (\partial_t u)^2(x,t) + |\nabla u|^2(x,t)\, dx$$

pomocný výpočet.

$$\frac{d}{dt}\int_{B(x_0,R-t)} g(x,t)\,dx = \left|\ \begin{array}{l} \text{substituce:} \\ x = x_0 + (R-t)y,\ y \in B(0,1) \\ dx = (R-t)^d\, dy \end{array}\right.$$

$$= \frac{d}{dt} \int_{B(0,1)} g(x_0 + (R-t)y, t)\,(R-t)^d \, dy$$

$$= \int_{B(0,1)} \frac{\partial}{\partial t} \left\{ g(x_0 + (R-t)y, t)\,(R-t)^d \right\} dy$$

$$= \int_{B(0,1)} \frac{\partial g}{\partial t}(x_0 + (R-t)y, t) \cdot (R-t)^d \, dy$$

$$+ \int_{B(0,1)} \nabla g(\dots) \cdot (-y)\,(R-t)^d + g(\dots)\,(-d)\,(R-t)^{d-1} \, dy$$

a nyní zpětná substituce:

$$= \int_{B(x_0, R-t)} \frac{\partial g}{\partial t}(x,t) \underbrace{- \nabla g(x,t) \cdot \frac{x-x_0}{R-t} - g(x,t)\frac{d}{R-t}}_{= -\operatorname{div}(g\,\underset{\sim}{w})} dy$$

kde $w = \dfrac{x-x_0}{R-t}$

(vzoreček: $\operatorname{div}(g w) = \nabla g \cdot w + g \operatorname{div} w$)

$g$ – skalár, $w$ – vektor

nyní dle Gausse: $\displaystyle\int_{B(x_0,R-t)} \operatorname{div}(g v)(x,t)\,dx$

$$= \int_{\partial B(x_0,R-t)} g(x,t)\, \underbrace{\frac{x-x_0}{R-t}\cdot n(t)}_{=1}\, d\sigma = \int_{\partial B(x_0,R-t)} g(x,t)\, d\sigma$$

(neboť $n(t) = \dfrac{x-x_0}{R-t}$ na $\partial B(x_0,R-t)$)

CELKEM: $$\frac{d}{dt}\int_{B(x_0,R-t)} g(x,t)\,dx = \int_{B(x_0,R-t)} \frac{\partial g}{\partial t}(x,t)\,dx - \int_{\partial B(x_0,t)} g(x,t)\,d\sigma$$

a tedy konečně:

$$e'(t) = \int_{B(x_0,R-t)} \frac{\partial}{\partial t}\left\{(\partial_t u)^2 + |\nabla u|^2\right\} dx - \int_{\partial B(x_0,R-t)} (\partial_t u)^2 + |\nabla u|^2\, d\sigma$$

$$= \int_{B(x_0,R-t)} 2\partial_t u\, \partial_{tt} u + \underbrace{2\nabla u \cdot \nabla \partial_t u}_{(*)}\, dx$$

$$- \int_{\partial B(x_0,R-t)} (\partial_t u)^2 + |\nabla u|^2\, d\sigma$$

opět dle Greena (viz důkaz V.III.1)

$$(*) = \int_{\partial B(x_0,R-t)} 2\frac{\partial u}{\partial n}\partial_t u\, d\sigma - \int_{B(x_0,R-t)} 2\Delta u\, \partial_t u\, dx$$

a tedy pokračujeme:

$$e'(t) = \int_{B(x_0,R-t)} 2\partial_t u\,(\underbrace{\partial_{tt} u - \Delta u}_{0})\, dx$$

$$- \int_{\partial B(x_0,R-t)} \underbrace{(\partial_t u)^2 + |\nabla u|^2 - 2\partial_t u \frac{\partial u}{\partial n}}_{\geq 0}\, d\sigma$$

neboť $\left|2\partial_t u \frac{\partial u}{\partial n}\right| \leq 2|\partial_t u||\nabla u|$, a dále Youngova ner.: $2|a||b| \leq a^2+b^2$

Věta III.3. [Fund. řešení (VR).]

dk.: cíl: najít řešení rce

$$\partial_{tt} u(x,t) - c^2 \Delta u(x,t) = \underbrace{\delta_{(0,0)}(x,t)}_{\delta_0(x)\,\delta_0(t)}$$

$\mathcal{F}_{x\to\xi}$, $t \in \mathbb{R}$ pevné:

$$\partial_{tt} \hat{u}(\xi,t) + \underbrace{4\pi^2 |\xi|^2 c^2}_{b^2,\ \text{kde } b = 2\pi|\xi|c} \hat{u}(\xi,t) = \delta_0(t)$$

Lemma II.2 (příklad 2)

$$\Rightarrow \hat{u}(\xi,t) = \underbrace{\frac{\sin(2\pi|\xi|ct)}{2\pi|\xi|c} \cdot Y(t)}_{\hat{H}(\xi,t)}$$

zbývá určit:

$$\mathcal{F}^{-1}_{\xi\to x}\left(\frac{\sin 2\pi|\xi|ct}{2\pi|\xi|c}\right), \quad \text{pro } t>0 \text{ pevné}$$

$d=1$, připomeň: $\mathrm{rect}(x) = \begin{cases} 1, & |x| < \frac{1}{2} \\ 0 & \text{jinde} \end{cases}$

(„čtvercová fce")

platí: $$\boxed{\widehat{\mathrm{rect}(x)} = \frac{\sin \pi|\xi|}{\pi|\xi|}, \quad (\xi \neq 0)}$$

a dále dle vzorce: $$\widehat{f\left(\frac{x}{a}\right)} = a\hat{f}(a\xi)$$ $$(a>0)$$

$$\Rightarrow \widehat{\mathrm{rect}\left(\frac{x}{a}\right)} = \frac{\sin \pi|\xi|a}{\pi|\xi|}, \quad \text{volme } \boxed{a = 2ct}$$

$$\frac{1}{2c}\widehat{\mathrm{rect}\left(\frac{x}{2ct}\right)} = \frac{\sin 2\pi|\xi|ct}{2\pi|\xi|c}$$

<u>a tedy konečně:</u>

$$H(x,t) = \frac{1}{2c}\underbrace{\mathrm{rect}\left(\frac{x}{2ct}\right)}_{= \begin{cases} 1, & |x| < ct \\ 0 & \text{jinde} \end{cases}} Y(t)$$

$$\left(\text{neboli } H(ct - |x|)\right)$$

$\underline{d=3}$, pomocný výpočet: $\widehat{\nu_r}(\xi) = ?$

nechť $\varphi(x) \in \mathcal{D}(\mathbb{R}^3)$ je pevné, libovolné:

$$\langle \widehat{\nu_r}, \varphi \rangle = \langle \nu_r, \widehat{\varphi} \rangle = \int_{|y|=r} \widehat{\varphi}(y)\, d\sigma(y)$$

$$= \int_{|y|=r} \left( \int_{\mathbb{R}^3} \varphi(x) e^{-2\pi i x\cdot y} dx \right) d\sigma(y)$$

$$= \int_{\mathbb{R}^3} \varphi(x) \underbrace{\left( \int_{|y|=r} e^{-2\pi i x\cdot y} d\sigma(y) \right)}_{I(x)} dx$$

$= \langle I, \varphi \rangle$, je tedy $I(x) = \widehat{\nu_r}(x)$

ve smyslu $\mathcal{D}'(\mathbb{R}^3)$.

zbývá spočítat $I(x)$, $x \in \mathbb{R}^3$ pevné

BÚNO: $x = (0, 0, \rho)$, kde $\rho = |x| > 0$

(neboť $I$ je radiální $\Leftarrow \nu_r$ radiální)

$$\Rightarrow I(x) = \int_{|y|=r} e^{-2\pi i \rho y_3}\, d\sigma(y) = \dots$$

$$y_1 = r\cos u \cos v$$
$$y_2 = r\sin u \cos v$$
$$y_3 = r\sin v$$
$$d\sigma = r^2 \cos v \, du\, dv$$

$\boxed{\text{sférické souřadnice}}$

$$u \in (0, 2\pi)$$
$$v \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\dots = \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-2\pi i \rho r \sin v}\, r^2 \cos v \, du\, dv$$

$$= 2\pi r \underbrace{\int_{-r}^{r} e^{-2\pi i \rho s}\, ds}_{= \frac{\sin 2\pi \rho r}{\pi \rho}}$$

substituce:

$$s = r \sin v$$
$$ds = r\cos v\, dv$$

$$= 2\pi r \frac{\sin(2\pi |x| r)}{\pi |x|}$$

výsledek lze např. pamatovat jako:

$$\underbrace{\frac{1}{4\pi r^2}}_{\text{povrch sféry}} \hat{V}_r(\xi) = \frac{\sin(2\pi|\xi| r)}{2\pi|\xi| r}$$

volbou $r = ct$ máme konečně:

$$\widehat{\frac{1}{4\pi c^2 t} V_{ct}(x)} = \frac{\sin(2\pi|\xi| ct)}{2\pi|\xi| c}$$

odkud plyne závěr.