Věta II.1. Nechť Ω je omezené, nechť $\partial_t u - \Delta u \leq 0$ v Ω_T.

Potom $\underbrace{\max_{\overline{\Omega}_T} u}_{M} = \underbrace{\max_{\Gamma_T} u}_{m}$.

Dk.

zřejmě $M \geq m$; ukážeme $M \leq m$.

KROK 1 nechť navíc $\partial_t u - \Delta u < 0$ v Ω_T

?? $M > m \Rightarrow \exists (x_0, t_0) \in \overline{\Omega}_T$

tak, že $u(x_0, t_0) = M > m$

nutně tedy $(x_0, t_0) \in \Omega_T \left(= \overline{\Omega}_T \setminus \Gamma_T\right)$.

dále buď (i) $\partial_t u(x_0, t_0) < 0$, nebo

(ii) $\Delta u(x_0, t_0) > 0$

ad(i) $\exists \tilde{t}_0 < t_0$ blízké t.ž. $(x_0, \tilde{t}_0) \in \Omega_T$,

a navíc $u(x_0, \tilde{t}_0) > u(x_0, t_0) = M$

(viz Věta z MAI) ↯↯

ad (ii) $\exists\ \tilde{x}_0$ blízké x_0 t.ž. $(\tilde{x}_0, t_0) \in \Omega_T$

nanic $u(\tilde{x}_0, t_0) > u(x_0, t_0) = M$

(viz důkaz Věty I.1) ↯↯

KROK 2 nechť obecně $\partial_t u - \Delta u \leq 0$

pomocné fce: $\tilde{u}(x,t) = u(x,t) - \varepsilon t$

($\varepsilon > 0$ libovolné, pevné)

$\Rightarrow \partial_t \tilde{u} - \Delta \tilde{u} = \partial_t u - \Delta u - \varepsilon < 0$ v Ω_T

dle K.1: $\max\limits_{\overline{\Omega_T}} \tilde{u} \leq \max\limits_{\Gamma_T} \tilde{u}$

avšak: $u = \tilde{u} + \varepsilon t \leq \tilde{u} + \varepsilon T$ v $\overline{\Omega_T}$

$\Rightarrow$ LS $\geq \left(\max\limits_{\overline{\Omega_T}} u\right) - \varepsilon T$

PS $\leq \max\limits_{\Gamma_T} u$

CELKEM: $\left(\max\limits_{\overline{\Omega_T}} u\right) - \varepsilon T \leq \max\limits_{\Gamma_T} u$

a pošleme $\varepsilon \to 0+$.

Věta II.2 nechť $\partial_t u - \Delta u \le 0$
v $\mathbb{R}^d \times (0,T]$, nechť u je shora omezené.
Potom $\sup\limits_{\mathbb{R}^d\times[0,T]} u = \sup\limits_{\mathbb{R}^d\times\{0\}} u$.

Dk. BÚNO PS $= 0$; cíl: $u(x,t) \le 0$
pro $\forall (x,t) \in \mathbb{R}^d \times (0,T]$.

pomocné fce: $\tilde{u} = u - \varepsilon v$, $\varepsilon > 0$

kde $v(x,t) = t + \dfrac{|x|^2}{2d}$.

vidíme: $\Delta |x|^2 = \Delta(x_1^2 + \dots + x_d^2) = 2d$

$\Rightarrow \partial_t v - \Delta v = 0$

nechť dále $R_0 > 0$ je t.ž. $\dfrac{\varepsilon R_0^2}{2d} \ge C$

kde $C = \sup\limits_{\overline{\Omega_T}} u < +\infty$

užiji Větu II.1. na fci $\tilde{u}$
pro $\Omega = B(0,R)$, $R \ge R_0$

$t=0: \tilde{u} = u \leq 0$

$|x|=R: \tilde{u} = u - \varepsilon v \leq C - \dfrac{\varepsilon R^2}{2d} < 0,$

nebo-li $R > R_0$

$\Rightarrow \tilde{u} \leq 0$ na $B(0,R) \times (0,T]$

(V: II.1 na $B(0,R)$)

libovolně velké

$\Rightarrow$ dostáváme $\mathbb{R}^d \times (0,T]$,

a konečně $\varepsilon \to 0+ \Rightarrow u \leq 0 \quad \forall x, t \geq 0$

<u>Věta II.3</u> [Energetická rovnost]

<u>Dk.</u> $\partial_t u(x,t) - \Delta u(x,t) = 0$

... „testuji" fcí u, tj. násobím $u(x,t)$

a integruji $\int_\Omega dx$

<u>1. člen</u>:

$$\int_\Omega \underbrace{\partial_t u(x,t) u(x,t)}_{\frac{1}{2}\frac{\partial}{\partial t} u^2(x,t)} dx = \frac{1}{2}\frac{d}{dt}\int_\Omega u^2(x,t)\,dx$$

2. člen: $\int_\Omega -\Delta u(x,t)\, u(x,t)\, dx$

$$= -\int_{\partial\Omega} \underbrace{\frac{\partial u}{\partial n}(x,t)\, u(x,t)}_{\substack{=0 \\ (\text{Dir.}/\text{Neum.})}} do + \int_\Omega \underbrace{\nabla u(x,t)\cdot \nabla u(x,t)}_{|\nabla u(x,t)|^2} dx$$

CELKEM

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2(x,t)\,dx + \int_\Omega |\nabla u(x,t)|^2 dx = 0 \quad \Big/ \int_0^\tau dt$$

$$\underbrace{\left[\frac{1}{2}\int_\Omega u^2(x,t)\,dx\right]_{t=0}^{t=\tau}}_{\shortparallel} + \int_0^\tau\int_\Omega |\nabla u(x,t)|^2 dx = 0$$

$$\frac{1}{2}\int_\Omega u^2(x,\tau)\,dx - \frac{1}{2}\int_\Omega \underbrace{u^2(x,0)}_{u_0^2(x)}\,dx$$

Lemma II.1 [Log-konvexita $e(t)$.]

Dk. z díky Věty II.3 víme:

$$e'(t) = 2\int_\Omega \partial_t u \cdot u = -2\int_\Omega \nabla u \cdot \nabla u \quad (*)$$

odkud pak:

$$e''(t) = -2\frac{d}{dt}\int_\Omega \nabla u \cdot \nabla u = -4\int_\Omega \underbrace{\partial_t \nabla u}_{\nabla \partial_t u} \cdot \nabla u$$

pomocný výpočet (Green):

$$\int_\Omega \nabla \partial_t u \cdot \nabla u = \underbrace{\int_{\partial\Omega} \partial_t u \cdot \frac{\partial u}{\partial n}}_{I_1} - \underbrace{\int_{\partial\Omega} \partial_t u \, \Delta u}_{I_2}$$

$I_1 = 0$, neboť na $\partial\Omega$ máme buď $\frac{\partial u}{\partial n} = 0$ nebo $u = 0$ ($\Rightarrow \partial_t u = 0$)

$I_2 = \int_\Omega (\partial_t u)^2$, neboť $\partial_t u = \Delta u$

odkud pak: $e''(t) = 4\int_\Omega (\partial_t u)^2 \quad (*)$

dále užij Hölderovu nerovnost:

$$\left|\int_\Omega fg\right| \le \left(\int_\Omega f^2\right)^{1/2}\left(\int_\Omega g^2\right)^{1/2}$$

$$\Rightarrow \quad \left(\int_\Omega fg\right)^2 \le \left(\int_\Omega f^2\right)\left(\int_\Omega g^2\right)$$

a tedy (viz (*) výše):

$$(e'(t))^2 = 4\left(\int_\Omega \partial_t u \cdot u\right)^2$$

$$\le 4\left(\int_\Omega \partial_t u\right)^2\left(\int_\Omega u^2\right) = e''(t)\cdot e(t),$$

tj. CELKEM: $\boxed{e'(t) \le e''(t)e(t)}$

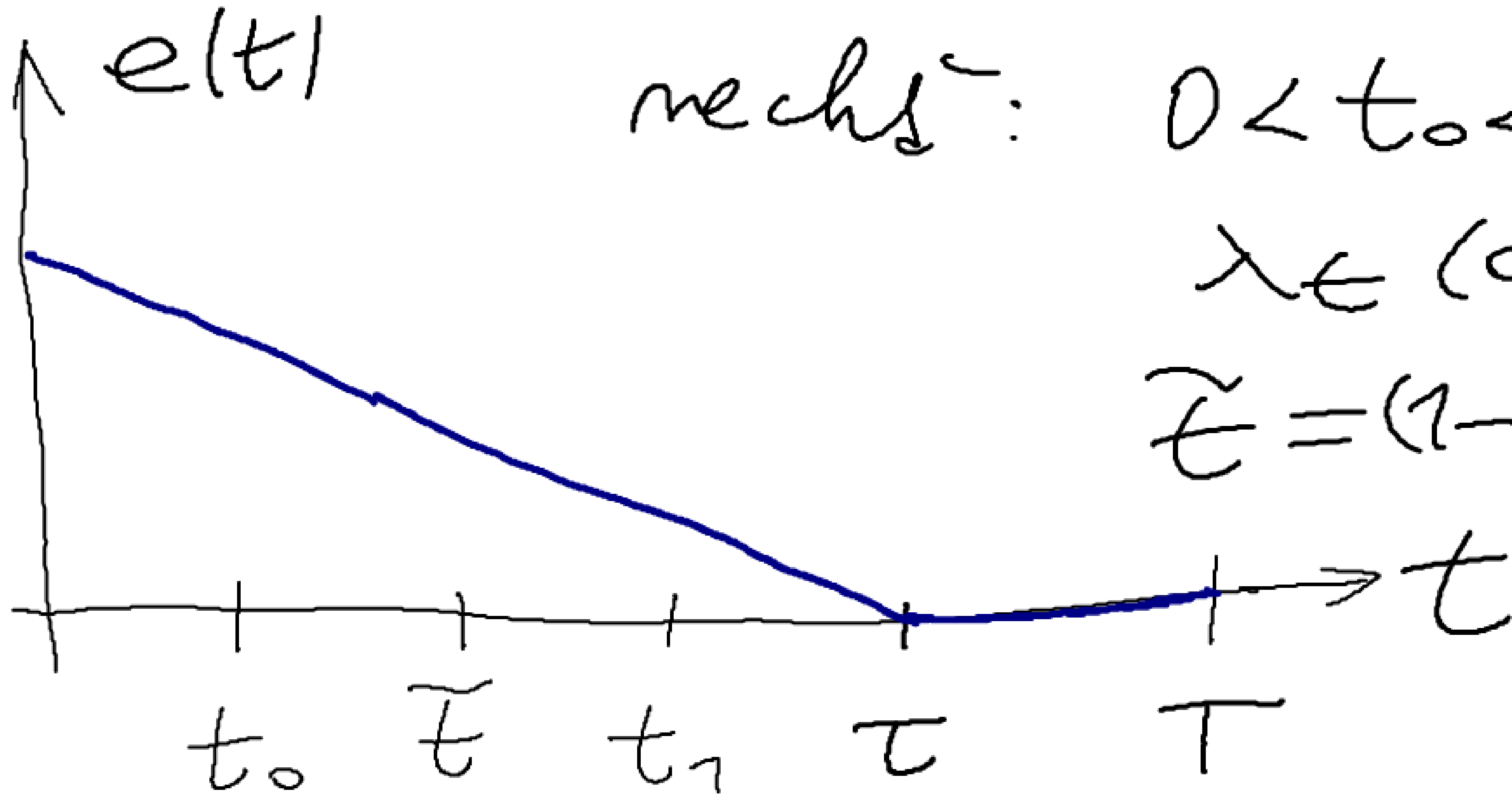
nyní pak již snadno:

$$(\log e(t))'' = \left(\frac{e'(t)}{e(t)}\right)' = \frac{e''(t)e(t)-(e'(t))^2}{e^2(t)} \ge 0$$

Důsledek (zpětná jednoznačnost)

dk. $e(T)=0 \overset{?}{\Longrightarrow} e(t)=0, \ \forall t\in[0,T)$

?? $\exists \tau \in (0,T)$ t.ž. $e(t) \begin{cases} =0, & t\geq\tau \\ >0, & t<\tau \end{cases}$



nechť: $0<t_0<t_1<\tau$

$\lambda\in(0,1)$

$\tilde{t}=(1-\lambda)t_0+\lambda t_1$

Lemma II.1. $\Rightarrow$ $\ell(t)=\log e(t)$

konvexní v $I=[0,\tau)$, tj:

$$\ell(\tilde{t}) \leq (1-\lambda)\ell(t_0)+\lambda\ell(t_1) \quad \Big| \exp$$

$$e(\tilde{t}) \leq e(t_0)^{1-\lambda}\cdot \underbrace{e(t_1)^{\lambda}}_{\to 0} \quad \Big| \ t_1\to\tau-$$

$\Rightarrow e(\tilde{t})=0, \ \tilde{t}<\tau \ \ldots$ ↯↯