Věta I.6 [Konformní záměna proměnné]

dk: $u(x,y) = v(\underbrace{f_1(x,y)}_{\xi}, \underbrace{f_2(x,y)}_{\eta}), \ (x,y) \in \Omega$

1./ $\partial_x u = \partial_\xi v \cdot \partial_x f_1 + \partial_\eta v \cdot \partial_x f_2$

$\partial_y u = \partial_\xi v \cdot \partial_y f_1 + \partial_\eta v \cdot \partial_y f_2$

$\Rightarrow \partial_x u + i \partial_y u$

$$= \partial_\xi v \cdot \underbrace{\left(\partial_x f_1 + i \underbrace{\partial_y f_2}_{-\partial_x f_2}\right)}_{\overline{f'}} + \partial_\eta v \cdot \underbrace{\left(\partial_x f_2 + i \underbrace{\partial_y f_2}_{\partial_x f_1}\right)}_{i\overline{f'}}$$

neboť $f' = \partial_x f_1 + i \partial_x f_2 = \partial_y f_2 - i \partial_y f_1$

dále $u \in C^1 \Leftarrow f_1, f_2, v \in C^1$

2. $\partial_{xx} u = \partial_x \left[\partial_\xi v \cdot \partial_x f_1 + \partial_\eta v \cdot \partial_x f_2 \right]$

$= \left(\partial_{\xi\xi} v \cdot \partial_x f_1 + \partial_{\eta\xi} v \cdot \partial_x f_2 \right) \cdot \partial_x f_1 + \partial_\xi v \cdot \partial_{xx} f_1$

$+ \left(\partial_{\xi\eta} v \cdot \partial_x f_1 + \partial_{\eta\eta} v \cdot \partial_x f_2 \right) \cdot \partial_x f_2 + \partial_\eta v \cdot \partial_{xx} f_2$

$\partial_{yy} u = \partial_y \left[\partial_\xi v \cdot \partial_y f_1 + \partial_\eta v \cdot \partial_y f_2 \right]$

$= \left(\partial_{\xi\xi} v \cdot \partial_y f_1 + \partial_{\xi\eta} v \cdot \partial_y f_2 \right) \cdot \partial_y f_1 + \partial_\xi v \cdot \partial_{yy} f_1$

$+ \left(\partial_{\xi\eta} v \cdot \partial_y f_1 + \partial_{\eta\eta} v \cdot \partial_y f_2 \right) \cdot \partial_y f_2 + \partial_\eta v \cdot \partial_{yy} f_2$

$\Longrightarrow \Delta_{x,y} u = \partial_{xx} u + \partial_{yy} u$

$= \partial_{\xi\xi} v \left[\left(\partial_x f_1 \right)^2 + \left(\partial_y f_1 \right)^2 \right]$

$+ \partial_{\eta\eta} v \left[\left(\partial_x f_2 \right)^2 + \left(\partial_y f_2 \right)^2 \right]$

$+ \partial_{\eta\xi} v \underbrace{\left[2 \partial_x f_2 \cdot \partial_x f_1 + 2 \partial_y f_2 \cdot \partial_y f_1 \right]}_{0}$

$+ \partial_\xi v \cdot \underbrace{\Delta f_1}_{0} + \partial_\eta v \cdot \underbrace{\Delta f_2}_{0} = \Delta_{\xi,\eta} v \cdot |\nabla f|$

3.) cíl $\Delta u = \delta_{f_{-1}(a)}$ v $\mathcal{D}'(\Omega)$, tj.

$$\langle \Delta u, \varphi \rangle = \langle \delta_{f_{-1}(a)}, \varphi \rangle \quad \forall \varphi \in \mathcal{D}(\Omega)$$

buď $\varphi \in \mathcal{D}(\Omega)$ dáno:

$$LS = \langle \Delta u, \varphi \rangle = \int_\Omega u \Delta \varphi \, dx dy$$

polož: $\psi = \varphi \circ f_{-1}$, tj. $\varphi = \psi \circ f$

dle bodu 1: $\Delta \varphi = \Delta \psi \circ f \cdot |\nabla f|$,

$$\text{tj.} \quad = \int_\Omega (v \circ \varphi) \cdot (\Delta \psi \circ f) \cdot |\nabla f| \, dx \, y$$

$$(V.o.S.) = \int_G v \Delta \psi \, d\xi dy = \langle \underbrace{\Delta v}_{\delta_a}, \psi \rangle$$

$$= \psi(a) = \varphi(f_{-1}(a)) = \langle \delta_{f_{-1}(a)}, \varphi \rangle = PS$$

zbývá ukázat

$$\boxed{\begin{array}{l} \varphi \in \mathcal{D}(\Omega) \Rightarrow \psi \in \mathcal{D}(G) \\ v \in L^1_{loc}(G) \Rightarrow u \in L^1_{loc}(\Omega) \end{array}}$$

ukážeme jen druhou implikaci:

cíl: $u \in L^1_{loc}(\Omega)$, tj: $\int_K |u| < +\infty$

pro $\forall$ kompakt $K \subset \Omega$

K dáno: polož $L := f(K) \subset G$

zřejmě L je kompakt

(spojitý obraz kompaktu)

dále jest $|Df| \geq c_0 > 0$ na K

(opět díky kompaktnosti, spojitosti)

TRIK: $$\int_K |u| = \int_K |v \circ f| \cdot \frac{|Df|}{|Df|} \leq c_0^{-1} \underbrace{\int_K |v \circ f| \cdot |Df|}_{\int_L |v| < +\infty,}$$

(neboť $v \in L^1_{loc}(G)$)

<u>Věta I.7</u> [O třech potenciálech.]

dk. BÚNO $x = 0$; tj. cílem je:

$$u(0) = \int_\Omega -\Delta u(y)\, \Phi(y)\, dy + \int_{\partial\Omega} \frac{\partial u}{\partial n}(y)\Phi(y) - u(y)\frac{\partial \Phi}{\partial n}(y)\, d\sigma(y)$$

Greenove identity:

$$\int_\Omega \Delta u \cdot v - u \cdot \Delta v = \int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot v - u \cdot \frac{\partial v}{\partial n}\, d\sigma$$

pro $\Omega_\varepsilon = \Omega \setminus B(0,\varepsilon)$, $v = \Phi$

$$\boxed{\Rightarrow \int_{\Omega_\varepsilon} \Delta u(y)\,\Phi(y) - u(y)\,\Delta\Phi(y)\,dy = \int_{\partial\Omega_\varepsilon} \frac{\partial u}{\partial n}(y)\,\Phi(y) - u(y)\,\frac{\partial \Phi}{\partial n}(y)\,d\sigma(y)}$$

LS $\longrightarrow \int_\Omega \Delta u(y)\,\Phi(y)\,dy$, neboť:

- $\Delta \Phi \equiv 0$ na $\Omega \setminus B(0,\varepsilon)$
- $\Delta u \cdot \Phi \in L^1(\Omega)$ ... Věta 18.9, bod 3

PS $\partial\Omega_\varepsilon = \partial\Omega \cup S(0,\varepsilon)$, a tedy

$$= \int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot \Phi - u \frac{\partial \Phi}{\partial n} + I_1 - I_2, \text{ kde}$$

$$I_1 = \int\limits_{S(0,\varepsilon)} \frac{\partial u}{\partial n} \Phi \, d\sigma \, , \quad I_2 = \int\limits_{S(0,\varepsilon)} u \frac{\partial \Phi}{\partial n} \, d\sigma .$$

ukážeme, že $\boxed{I_1 \to 0, \; I_2 \to u(0).}$

$$|I_1| \le \int\limits_{S(0,\varepsilon)} |\nabla u| \, |\Phi| \, d\sigma \le C_1 \Pi_\varepsilon \, \sigma(S(0,\varepsilon))$$

kde $|\nabla u| \le C_1$ na okolí 0 (spojitost)

$$\Pi_\varepsilon = \max_{y \in S(0,\varepsilon)} |\Phi| \le \begin{cases} C_2 |\ln \varepsilon| , & d=2 \\ C_2 |\varepsilon|^{2-d} , & d \geq 3 \end{cases}$$

$$\sigma(S(0,\varepsilon)) = \beta_d \, \varepsilon^{d-1}$$

$$\Rightarrow |I_1| \le \begin{cases} C \cdot \varepsilon |\ln \varepsilon| & , d=2 \\ C \cdot \varepsilon & , d \geq 3 \end{cases}$$

a tedy $|I_1| \to 0 , \; \varepsilon \to 0+$

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ad $\underset{\smile}{I_2}$, $\nabla \Phi(y) = \dfrac{-y}{\beta_d |y|^d} , \; n(y) = \dfrac{-y}{|y|}$

a sedy $\frac{\partial \Phi}{\partial n} = \nabla \Phi(y) \cdot n(y) = \frac{1}{\beta_d |y|^{d-1}}$

(pro $|y| = \varepsilon$) $\quad = \frac{1}{\beta_d \varepsilon^{d-1}} = \frac{1}{\sigma(S(0,\varepsilon))}$

$\Rightarrow I_2 = \int_{S(0,\varepsilon)} u \, d\sigma \to u(0), \ \varepsilon \to 0+$

(Lemma I.1)

<u>Věta I.8</u> [O Greenově funkci.]

<u>dk.</u> dle V. I.7 máme:

$$u(x) = \int_\Omega \underbrace{-\Delta u(y)}_{f(y)} \Phi(y) \, dy + \int_{\partial\Omega} \frac{\partial u}{\partial n}(y) \Phi(y-x)$$

$$- \underbrace{u(y)}_{h(y)} \frac{\partial \Phi}{\partial n}(y-x) \, d\sigma(y)$$

sedy celkem:

$$\boxed{u(x) = \int_\Omega f(y) \Phi(y-x) \, dy + \int_{\partial\Omega} \frac{\partial u}{\partial n}(y) \Phi(y-x) - h(y) \frac{\partial \Phi}{\partial n}(y-x) \, d\sigma(y) \quad (+)}$$

dále, pomocí Greenovy ident.:

$$\int_\Omega \underbrace{\Delta u}_{= -f} \cdot \phi^x - u \cdot \Delta\phi^x = \int_{\partial\Omega} \frac{\partial u}{\partial n} \underbrace{\phi^x}_{= \Phi(\cdot - x)} - \underbrace{u}_{= h} \frac{\partial \phi^x}{\partial n}\, d\sigma$$

sečteme celkem:

$$\boxed{0 = \int_\Omega f(y)\phi^x(y)\,dy + \int_{\partial\Omega} \frac{\partial u}{\partial n}(y)\,\Phi(y-x) - u(y)\frac{\partial \phi^x}{\partial n}(y)\,d\sigma(y) \qquad (++)}$$

Odečtením (+) a (++) plyne řešení.