Exercises on AC solutions.

1. Assume $x(t)$, $y(t) \in AC(I)$, where I is compact interval, and $\phi : \mathbb{R} \to \mathbb{R}$ is C^1. Prove that:
   (i) $x(t)y(t) \in AC(I)$ and $(x(t)y(t))' = \dots$
   (ii) if $x(t) \neq 0$, then also $1/x(t) \in AC(I)$ and $(1/x(t))' = \dots$
   (iii) $\phi(x(t)) \in AC(I)$ and $\phi(x(t))' = \dots$

2. Assume $x(t) \in AC(I)$. Prove that $|x(t)| \in AC(I)$ and $|x(t)|' = \operatorname{sgn}(x(t))x'(t)$ almost everywhere.

   Hint: approximate $|y|$ with $|y|_\varepsilon = \sqrt{y^2 + \varepsilon}$. Write $|x(t_1)|_\varepsilon - |x(t_2)|_\varepsilon = \int_{t_1}^{t_2} \big(\dots\big)' \, dt$ and take $\varepsilon \to 0+$.

   Optionally: state and prove the corresponding statement for $x(t)$ vectorial, i.e. $x(t) : I \to \mathbb{R}^n$.

3. Assume $a(t)$, $b(t) \in L^1_{loc}(I)$. Then $x(t)$ is a Caratheodory solution to

$$x' + a(t)x = b(t) \tag{1}$$

   on I if and only if $x(t) = e^{-A(t)}\big(c + \int e^{A(s)} b(s)\, ds\big)$, where $A(t) = \int a(t)\, dt$.

4. Function $F(x) : \mathbb{R}^n \to \mathbb{R}^n$ is called *monotone*, if $(F(x) - F(y)) \cdot (x - y) \geq 0$ for all x, $y \in \mathbb{R}^n$.

   Prove that for any $x(t)$, $y(t) \in AC_{loc}$ which satisfy

$$x' + F(x) = 0 \tag{2}$$

   for almost all t, one has $|x(t) - y(t)| \leq |x(t_0) - y(t_0)|$, for all $t \geq t_0$. In particular, the equation (2) has the property of *forward* uniqueness.

   Hint: multiply the equation for $u = x - y$ by u and integrate.