Theorem 19.6 [Hopf in $\mathbb{R}^m$] Consider (Ha) $x' + f(t,\mu) = 0$,

$f: \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ is C^2, $f(0,\mu) = 0$ close to $(0,0) \in \mathbb{R}^m \times \mathbb{R}$.

Denote $A_\mu = D_x f(0,\mu) \in \mathbb{R}^{m\times m}$. Assume that

1. $\pm i\omega_0 \in \sigma(A_\mu)$ simple, but $\pm i k \omega_0 \notin \sigma(A_\mu)$ $\forall k \geq 2$
2. $\exists\, C^1$ curves $\mu \mapsto a(\mu) \in \mathbb{C}^m$, $\mu \mapsto \beta(\mu) \in \mathbb{C}$ such that
   $A_\mu a(\mu) = \beta(\mu) a(\mu)$, $a(0) \neq 0$, $\beta(0) = i\omega_0$, $\operatorname{Re}\beta'(0) \neq 0$.

Then: $\exists$ a C^1 curve $\rho \longmapsto (x_\rho, \rho_\rho, \mu_\rho)$ to a neighborhood of $(\mathbb{0}, \frac{1}{\omega_0}, 0) \in C^1 \times \mathbb{R} \times \mathbb{R}$ s.t. for $\forall \rho \neq 0$, x_ρ is a non-trivial $2\pi\rho_\rho$-periodic solution of (Ha), with $\mu = \mu_\rho$.

Moreover: there are no other periodic (non-trivial) solutions close to $(\mathbb{0}, \frac{1}{\omega_0}, 0)$.

Pf. WLOG $\omega_0 = 1$; observe : $x(t)$ is $2\pi\rho$-periodic sol. to (Ha)

$$\Updownarrow$$

$u(\tau) = x(\rho\tau)$ is 2π-per. sol. to (Hb) $u' + \rho f(u,\mu) = 0$.

$\Rightarrow$ abstract bifurcation $F(u, \underbrace{\rho, \mu}_{\lambda \in \mathbb{R}^2 \text{ -- parameter}}) = \mathbb{0}$

where $F: X \times \mathbb{R} \times \mathbb{R} \to Y$

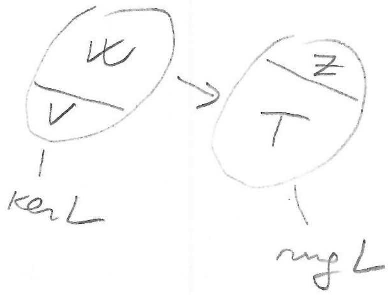
$$(u, \rho, \mu) \longmapsto u' + \rho f(u,\mu)$$

$$X = C^1_{2\pi},\ Y = C_{2\pi}.$$

.. close to $(\mathbb{0}, 1, 0)$

Set $\quad L = F_u(0,1,0) : \phi \longmapsto \phi' + A_0 \phi$

Apply Lemma 19.2 (and its notation)



i.e. $L : W \to T$ isomorph.

$V = \ker L = \operatorname{span}\{\phi_0, \phi_1\}$

$Z = (\operatorname{rng} L)^\perp = \operatorname{span}\{\psi_0, \psi_1\} = \ker L^*$

Set $\quad \underset{\substack{\uparrow \\ W}}{g(v}, \beta, \mu, \rho) := F(\underbrace{\rho(\phi_0 + v)}, \beta, \mu)$

new param. (amplitude) $\qquad$ Ansatz for u (= sought for periodic solution)

L.19.1 (division l.) $\quad \ldots \quad g(v,\beta,\mu,\rho) = \rho\, G(v,\beta,\mu,\rho)$

strategy: apply IFT to $G(v,\beta,\mu,\rho) = 0$ to get

$$\rho \longmapsto (v_\rho, \beta_\rho, \mu_\rho) \text{ a curve of sol.}$$

$$\Rightarrow u = u_\rho = \rho(\underset{\substack{\uparrow \\ V}}{\phi_0} + \underset{\substack{\uparrow \\ W}}{v_\rho}) \neq 0, \ \rho \neq 0$$

---

compute: $\quad G(v,\beta,\mu,0) = g_\rho(v,\beta,\mu,0)$

$$= F_u(0,\beta,\mu)$$

$$= (\phi_0 + v)' + \beta A_\mu (\phi_0 + v)$$

$$G(0,1,0,0) = \phi_0' + 1 \cdot A_0 \phi_0 = 0 \in Y$$

remains to show: $\quad M := G_{v,\beta,\mu}\Big|_{(0,1,0,0)} : W \times \mathbb{R} \times \mathbb{R} \longrightarrow Y$

isomorphism ?

denote $\mathbb{1} = (0,1,0,0)$

compute: $G_v(\mathrm{II}) = g_{\sigma v}(\mathrm{II})v = v' + A_0 v$
(L.19.1)
$G_\rho(\mathrm{II}) = g_{\sigma\rho}(\mathrm{II}) = A_0\phi_0$
$G_\mu(\mathrm{II}) = g_{\sigma\mu}(\mathrm{II}) = B_0\phi_0$; $B_0 = \frac{d}{d\mu} A_\mu \Big|_{\mu=0}$

---

hence: $\Gamma : (v, r, m) \longmapsto \underbrace{(v' + A_0 v)}_{\in T} + \underbrace{r A_0\phi_0 + m B_0\phi_0}_{\in Z}$

but: $v \longmapsto v' + A_0 v$
$W \longrightarrow Z$
$\Big\}$ .. isomorphism

remains to show: $(r, m) \longmapsto r A_0\phi_0 + m B_0\phi_0$
$\mathbb{R}^2 \longrightarrow T$ .. also isomorphism

---

solve $r A_0\phi_0 + m B_0\phi_0 = R$, $R \in Z$ given

$\Updownarrow$

$\langle r A_0\phi_0 + m B_0\phi_0 - R, \psi_j \rangle = 0$, $j = 0,1$

$\Updownarrow$ $\begin{pmatrix} \mathcal{A} \end{pmatrix} \begin{pmatrix} r \\ m \end{pmatrix} - \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = 0$

where $R_j = \langle R, \psi_j \rangle$, $\mathcal{A} = \begin{pmatrix} \langle A_0\phi_0, \psi_0 \rangle, \langle B_0\phi_0, \psi_0 \rangle \\ \langle A\phi_0, \psi_1 \rangle, \langle B_0\phi_0, \psi_1 \rangle \end{pmatrix}$

but by L.19.2: $\mathcal{A} = \begin{pmatrix} 0 & c \\ 1 & d \end{pmatrix}$, $c = \langle B_0\phi_0, \psi_0 \rangle \neq 0$

regular .. $\boxed{\text{done}}$.

Lemma 19.2 Assume: $A_0 \in \mathbb{R}^{m\times m}$, $\pm i \in \sigma(A_0)$ simple, $\pm ik \notin \sigma(A_0)$ $\forall k \geq 2$. Denote $L: \phi \longmapsto \phi' + A_0\phi$

$L^*: \psi \longmapsto -\psi' + A_0^*\psi$.

Then: 1. $\exists\, \phi_0, \phi_1 \in C^1_{2\pi}$ s.t. $\ker L = \operatorname{span}\{\phi_0, \phi_1\}$, and

$\phi_0' = \phi_1$, $\phi_1' = -\phi_0$

2. $\exists\, \psi_0, \psi_1 \in C^1_{2\pi}$ s.t. $\ker L^* = (\operatorname{rng} L)^\perp = \operatorname{span}\{\psi_0, \psi_1\}$, and

$\langle \phi_i, \psi_j \rangle = 2\pi\,\delta_{ij}$, $\forall i,j = 0,1$

3. If A_μ as in Thm 19.6, $A_0 = A_\mu|_{\mu=0}$, $B_0 = \frac{d}{d\mu} A_\mu\Big|_{\mu=0}$,

then $\langle B_0\phi_0, \psi_0 \rangle \neq 0$.

Proof. change of vars: $A_0 = P\tilde{A}_0P^{-1}$; $\phi = P\tilde{\phi}$:

$(P \in \mathbb{C}^{m\times m}$ reg.$)$

$\phi' + A_0\phi = 0 \iff \tilde{\phi}' + \tilde{A}_0\tilde{\phi} = 0$.

$A_0^* = (P^{-1})^*\tilde{A}_0^*P^*$, $\psi = P^*\tilde{\psi}$

$-\psi' + A_0^*\psi = 0 \iff -\tilde{\psi}' + \tilde{A}_0^*\tilde{\psi} = 0$

also note: $\langle \tilde{\phi}, \tilde{\psi} \rangle = \langle P^{-1}\phi, (P^{-1})^*\psi \rangle = \langle \phi, \underbrace{P^*(P^{-1})^*}_{= I}\psi \rangle$.

$\Rightarrow \boxed{\text{WLOG}}$: $A_0 = \left(\begin{array}{cc|c} 0 & 1 & \\ -1 & 0 & 0 \\ \hline 0 & & \lambda_\cdot \neq \pm ik,\ k \geq 2 \end{array}\right)$

ker L = ? ... find $\phi \in C^1_{2\pi}$: $\phi' + A_0\phi = 0$

$L\phi = 0$ $\qquad$ $\phi(0) = \phi(2\pi)$.

— $\phi(t) = e^{-tA_0}\omega$; $\omega \in \mathbb{C}^m$; $\left(e^{-2\pi A_0} - I\right)\omega = 0$

but. $$e^{tA_0} = \left(\begin{array}{cc|c} \cos t, & \sin t & 0 \\ -\sin t, & \cos t & \\ \hline 0 & & e^{\lambda t} \; \ddots \quad \lambda \neq \pm i \mathbb{Z} \end{array}\right)$$

$$\Rightarrow e^{-2\pi A_0} = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & \\ \hline 0 & & e^{-2\pi\lambda} \neq 1 \; \ddots \end{array}\right)$$

$\Rightarrow \left(e^{-2\pi A_0} - I\right)\omega = 0$ iff $\omega_j = 0$, $j > 2$.

... take $\omega = (1, 0 ——)$

$(0, 1, 0 ——)$

$\Rightarrow \phi_0 = (\cos t, \sin t, 0 ——)$

$\phi_1 = (\sin t, \cos t, 0 ——)$.

ker L^* = ? ... analogous argument : $\psi = e^{tA_0^*}\eta$

$\Rightarrow \psi_0 = (\cos t, \sin t, 0 ——) = \phi_0$

$\psi_1 = (-\sin t, \cos t, 0 ——) = \phi_1$

$\operatorname{rng} L = ?$ given $h \in C_{2\pi}$, find $\phi \in C^1_{2\pi}$ s.t.

$$\phi' + A_0 \phi = h$$
$$\phi(0) = \phi(2\pi).$$

Var of const: $\phi(t) = e^{-tA_0}\left(\omega + \int_0^t e^{sA_0} h(s)\, ds\right)$

$$\phi(0) = \phi(2\pi) \quad \cdots \quad \underbrace{\left(e^{-2\pi A_0} - I\right)}_{M} \omega = -e^{-2\pi A_0} \cdot \underbrace{\int_0^{2\pi} e^{sA_0} h(s)\, ds}_{b}$$

$$e^{-2\pi A_0} = \left(\begin{array}{c|c} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0 \\ \hline 0 & \text{regular} \end{array}\right); \quad M = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \text{reg.} \end{array}\right)$$

i.e. problem is solvable iff $b_j = 0 \ \forall j > 2$

$$0 = b \cdot \eta = \int_0^{2\pi} e^{sA_0} h(s) \cdot \eta \, ds = \int_0^{2\pi} h(s) \cdot e^{sA_0^*} \eta \, ds$$

$$= \langle h, \underbrace{e^{sA_0^*} \eta}_{} \rangle; \quad \forall \eta \in \mathbb{C}^n, \ \eta_k = 0; \ k > 2$$

general element of $\ker L^*$.

---

finally: $\beta(\mu) \in \sigma(A_\mu)$; $\beta(\mu) = \alpha(\mu) \pm i\omega(\mu)$

$$A_\mu = \left(\begin{array}{c|c} \begin{matrix} \alpha(\mu), & \omega(\mu) \\ -\omega(\mu), & \alpha(\mu) \end{matrix} & \dots \\ \hline & \end{array}\right); \quad B_0 = \left(\begin{array}{c|} \begin{matrix} \alpha'(0), & \omega'(0) \\ -\omega'(0), & \alpha'(0) \end{matrix} \\ \hline \end{array}\right)$$

$$\langle B_0 \phi_0, \psi_0 \rangle = (\cos t, \sin t) \begin{pmatrix} \alpha'(0), & \omega'(0) \\ -\omega'(0), & \alpha'(0) \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} =$$

$$= \int_0^{2\pi} \alpha'(0) \left[\cos^2 t + \sin^2 t\right] + \omega'(0) \left[\cos t \sin t - \cos t \sin t\right] dt$$

$$= 2\pi \alpha'(0) \neq 0.$$