19. Bifurcation theory

Def. [Bifurcation – ODE]. (x_0,λ_0) ... regular point of

(19.1) $\quad x' = f(x,\lambda)$ :: $\exists\, \delta>0$, $\mathcal{U}$ neigh. of x_0 s.t. dyn. sys. of (19.1) are top. conjugate in $\mathcal{U}$.

(x_0,λ_0) .. point of bifurcation $\Leftrightarrow$ is not regular.

Rem. dyn. syst. $x'=f(x)$ in $\mathcal{U}$, $x'=\tilde{f}(x)$ in $\tilde{\mathcal{U}}$ top. conj. ::

$\exists$ homeomorphism $h:\mathcal{U}\to\tilde{\mathcal{U}}$ s.t. $x(t)$ solves (i) in $\mathcal{U}$ $\Leftrightarrow$ $h(x(t))$ solves (ii) in $\tilde{\mathcal{U}}$.

Rem. ① $f(x_0,\lambda_0)\neq 0 \Rightarrow (x_0,\lambda_0)$ regular. by rectification lemma (Thm 13.3)

[unif. w.r.to $|\lambda-\lambda_0|<\delta$

d.s. top. conj. to $y' = \begin{pmatrix}1\\0\\0\end{pmatrix}$. $\quad x_0\in\mathcal{U}$

② $f(x_0,\lambda_0)=0$, but $\operatorname{Re}\lambda\neq 0\ \forall \lambda\in\sigma(A)$

$$A = \nabla_x f(x_0,\lambda_0)$$

("hyperbolic stat. point") $\Rightarrow (x_0,\lambda_0)$ .. regular.

Pf. (i) consider $f(x,\lambda)=0$ close to (x_0,λ_0)

by IFT $\exists!$ sol. $x=\hat{x}(\lambda)$; $x_0=\hat{x}(\lambda_0)$

$\left(A=\nabla_x f(x_0,\lambda_0) \text{.. regular, since } 0\notin\sigma(A)\right)$.

(ii) $\sigma(A_\lambda)$ .. "close to" $\sigma(A)$, $|\lambda-\lambda_0|<\delta$

$A_\lambda = \nabla_x f(\hat{x}(\lambda),\lambda)$ .. "hyperbolic stationary"

(iii) d.s. top. equival. to $y' = A_\lambda y$

Cor. Necessary condition for bifurcation: non-hyperbolic, stationary point; i.e. $f(x_0, \lambda_0) = 0$

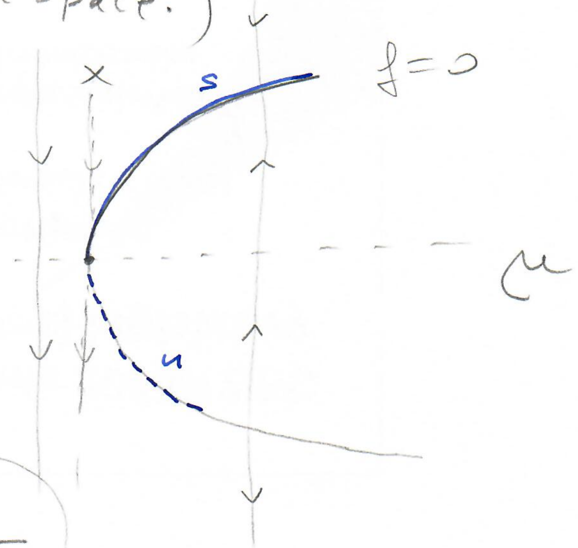
$$\sigma(D_x f(x_0, \lambda_0)) \cap i\mathbb{R} \neq \phi.$$

Plan:
- simple bif. in $\mathbb{R}$
- Hopf bif. in $\mathbb{R}^2$
- abstract bif. (in Banach space.)

Ex. 1 $x' = \mu - x^2 = f(x, \mu)$

$x \in \mathbb{R}$

bif-parameter



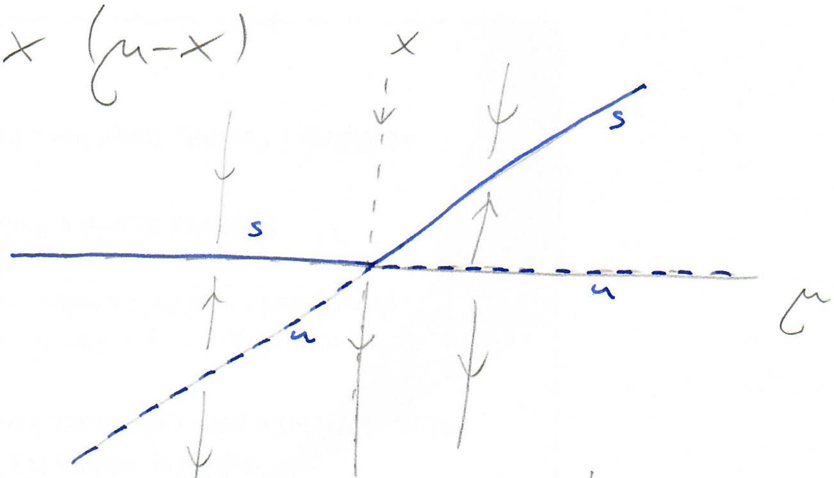
$(0,0) \cdots$ the only bif. point.

"saddle-node"

story

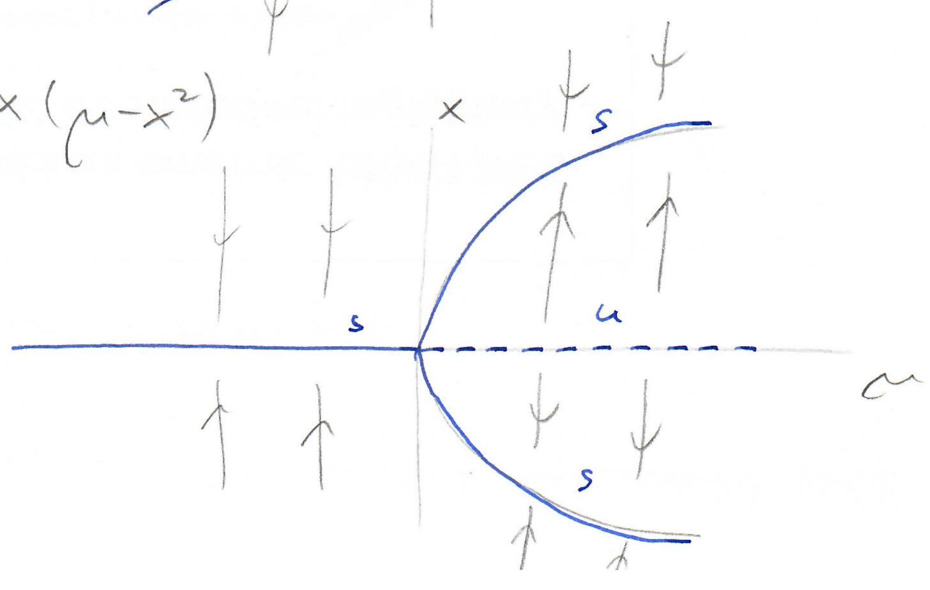
Ex. 2 $x' = \mu x - x^2 = x(\mu - x)$

"trans-critical bif."



Ex. 3 $x' = \mu x - x^3 = x(\mu - x^2)$

"pitchfork bif."



Lemma 19.1 [Division Lemma.]

Let $h(x,\lambda) \in C^k$, $h(0,\lambda)=0$ close to $(0,0) \in \mathbb{R}^2$

Then $\exists\, H(x,\lambda) \in C^{k-1}$ s.t. $h(x,\lambda) = x H(x,\lambda)$ close to $(0,0)$.

Moreover: $H(0,0) = \partial_x h(0,0)$

$$\partial_x H(0,0) = \tfrac{1}{2}\partial^2_{xx} h(0,0)$$

$$\partial_\lambda H(0,0) = \partial^2_{x\lambda} h(0,0)$$

$$\partial^2_{xx} H(0,0) = \tfrac{1}{3}\partial^3_{xxx} h(0,0)$$

Pf. Set $H(x,\lambda) = \int_0^1 (\partial_x h)(x\sigma,\lambda)\,d\sigma$

$$x H(x,\lambda) = \int_0^1 \underbrace{x\,\partial_x h(x\sigma,\lambda)}_{\partial_\sigma h(x\sigma,\lambda)}\,d\sigma = \Big[h(x\sigma,\lambda)\Big]_{\sigma=0}^{\sigma=1}$$

$$= h(x,\lambda) - \underbrace{h(0,\lambda)}_{=0}$$

$H(x,\lambda)$ .. continuous .. (C^0) .. easy

$$\partial_x H(x,\lambda) = \partial_x \int_0^1 \partial_x h(x\sigma,\lambda)\,d\sigma = \int_0^1 \sigma\,\partial^2_{xx} h(x\sigma,\lambda)\,d\sigma$$

$\Rightarrow \partial_x H$ .. cont. ; $\partial_x H(0,0) = \int_0^1 \sigma\,\partial^2_{xx} h(0,0)\,d\sigma$

$$= \tfrac{1}{2}\partial^2_{xx} h(0,0)$$

$$\partial_\lambda H(x,\lambda) = \partial_\lambda \int_0^1 \partial_x h(x\sigma,\lambda)\,d\sigma = \int_0^1 \partial^2_{x\lambda} h(x\sigma,\lambda)\,d\sigma$$

$\Rightarrow \partial_\lambda H$ .. cont., $\partial_\lambda H(0,0) = \int_0^1 \partial^2_{x\lambda} h(0,0)\,d\sigma$

$k=1$ proven ; $k \geq 2$ similar. $= \partial^2_{x\lambda} h(0,0)$

Theorem 19.1 [Saddle-node 1d.]

Assume: $f(x,\mu) \in C^2$ close to $(0,0) \in \mathbb{R}^2$

$$f(0,0)=0, \quad \partial_x f(0,0)=0$$

$$\partial_\mu f(0,0) \neq 0, \quad \partial^2_{xx} f(0,0) \neq 0$$

Then: equation (19.2) $x' = f(x,\mu)$ has a saddle-node bif. in $(0,0)$.

Pf. consider eq. $f(x,\mu)=0$

$f(0,0)=0$, $\partial_\mu f(0,0) \neq 0$: by IFT $\exists\, \hat\mu = \hat\mu(x)$

$$\hat\mu : \mathcal{U}(0,\delta) \to \mathcal{U}(0,\Delta)$$

s.t. $f(x,\mu)=0$ in $\mathcal{U}(0,\delta) \times \mathcal{U}(0,\Delta)$ $\quad \hat\mu(0)=0$

$\iff$ $\mu = \hat\mu(x)$.

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we know: $\hat\mu(0)=0$, $\hat\mu'(0)=0$, $\hat\mu''(0) \neq 0$.

$f(x,\hat\mu(x))=0$, $x \in \mathcal{U}(0,\delta)$ $\quad \frac{d}{dx}$

$$\partial_x f(x,\hat\mu(x)) + \partial_\mu f(x,\hat\mu(x))\hat\mu'(x)=0, \quad x=0$$

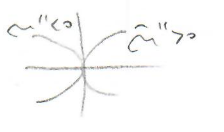
$$\partial_x f(0,0) + \partial_\mu f(0,0)\cdot \hat\mu'(0)=0$$

$$\hat\mu'(0) = \frac{-\partial_x f(0,0)}{\partial_\mu f(0,0)} = 0.$$

$\frac{d^2}{dx^2}$: $$f_{xx} + f_{x\mu}\cdot \hat\mu' + \partial_{\mu x} f\, \hat\mu' + \partial_{\mu\mu} f (\hat\mu')^2 + \partial_\mu f\, \hat\mu'' = 0$$

$x=0$. $$f_{xx}(0,0) + \partial_\mu f(0,0)\hat\mu''(0) = 0$$

picture



$$\hat\mu''(0) = \frac{f_{xx}(0,0)}{\partial_\mu f(0,0)} \neq 0$$

Theorem 19.2 [Transcritical 1d.]

Assume: $f(x,\mu) \in C^2$ close to $(0,0) \in \mathbb{R}^2$

$f(0,0) = 0, \ \partial_x f(0,0) = 0$

$f(0,\mu) = 0$ close to 0,

$\partial^2_{\mu x} f(0,0) \neq 0, \ \partial^2_{xx} f(0,0) \neq 0.$

Then: (19.2) has a transcritical bif. in $(0,0)$.

Pf. by Lemma 19.1 (division L.): $f(x,\mu) = x F(x,\mu)$; $F \in C^1$ close to $(0,0)$.

$f(x,\mu) = 0 \iff x = 0 \vee \boxed{F(x,\mu) = 0.}$

$F(0,0) = \partial_x f(0,0) = 0$

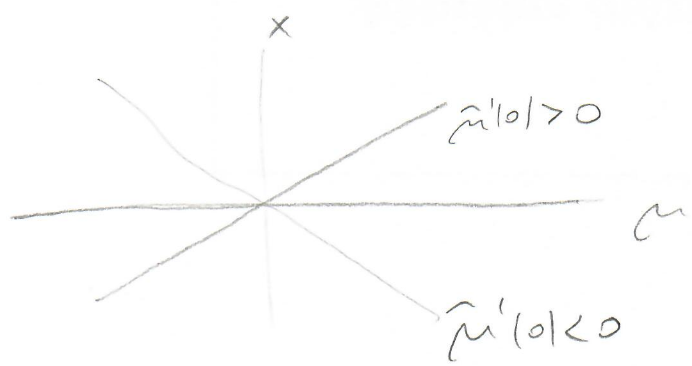
$\partial_\mu F(0,0) = \partial^2_{x\mu} f(0,0) \neq 0$

$\therefore$ IFT: $F(x,\mu) = 0 \iff \mu = \hat{\mu}(x) \in C^1$

$\hat{\mu}(0) = 0.$

$\hat{\mu}'(0) = ?$ as before $\hat{\mu}'(0) = \dfrac{-\partial_x F(0,0)}{\partial_\mu F(0,0)} \neq 0$

since $\underset{\substack{\uparrow \\ \text{L. 19.1}}}{\partial_x F(0,0) =} \frac{1}{2} \underset{\substack{\uparrow \\ \text{assumption}}}{\partial^2_{xx} f(0,0) \neq 0}$



Theorem 19.3 [Pitchfork 1d.]

Assume: $f(x,\mu) \in C^3$ close to $(0,0) \in \mathbb{R}^2$

$f(0,0)=0,\ \partial_x f(0,0)=0$

$f(0,\mu)=0$ close to 0, $\partial_{xx} f(0,0)=0$

$\partial^2_{x\mu} f(0,0) \neq 0,\ \partial^3_{xxx} f(0,0) \neq 0$

Then: (19.2) $x' = f(x,\mu)$ has a pitchfork bifurc. in $(0,0)$.

Pf. consider $f(x,\mu)=0$ ... by Division Lemma (L.19.1)

$$f(x,\mu) = x\, F(x,\mu);\quad F \in C^2$$

moreover: $F(0,0) = \partial_x f(0,0) = 0$

$\partial_\mu F(0,0) = \partial^2_{x\mu} f(0,0) \neq 0 \quad \overset{\text{IFT}}{\Longrightarrow} \quad F(x,\mu)=0$

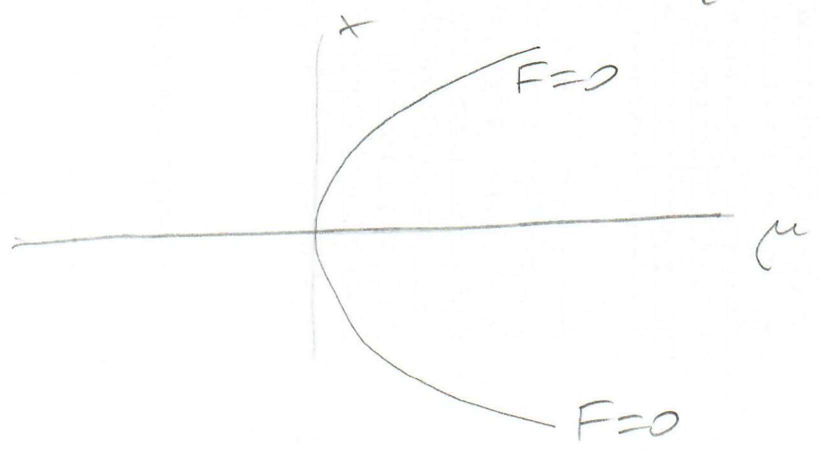
$\Leftrightarrow \quad \mu = \hat{\mu}(x)$

.. properties of $\hat{\mu}(x)$ ... as in Th. 19.1

(F instead of f)

$$\hat{\mu}'(0) = \frac{-\partial_x F(0,0)}{\partial_\mu F(0,0)} = -\frac{\partial^2_{xx} f(0,0)}{\partial^2_{x\mu} f(0,0)} = 0$$

$$\hat{\mu}''(0) = \frac{\partial_{xx} F(0,0)}{\partial_\mu F(0,0)} = \frac{\frac{1}{3}\partial^3_{xxx} f(0,0)}{\partial^2_{x\mu} f(0,0)} \neq 0$$



complete picture (sign of f, F) by sign of non-zero terms

Theorem 19.4 [Hopf bif. in 2d]

Consider system (19.3) $\begin{pmatrix} x \\ y \end{pmatrix}' = A_\mu \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(x,y,\mu) \\ g(x,y,\mu) \end{pmatrix}$.

Assume: $f, g, \nabla_x f, \nabla_x g$ smooth, zero in $(0,0,\mu)$.

$A_\mu \in \mathbb{R}^{2\times 2}$, smooth w.r.t. μ, and

key assump.: $\sigma(A_\mu) = \{\alpha(\mu) \pm i\,\omega(\mu)\}$

$\alpha(0) = 0, \ \alpha'(0) \neq 0, \ \omega(0) \neq 0.$

Then: $\exists$ family of (non-trivial) periodic sol. close to $(x, y, \mu) = (0, 0, 0)$.

Pf. WLOG $A_\mu = \begin{pmatrix} \alpha(\mu), & -\omega(\mu) \\ \omega(\mu), & \alpha(\mu) \end{pmatrix}$ .. by linear change of coordinates

i.e.
$$x' = \alpha(\mu)x - \omega(\mu)y + f(x,y,\mu)$$
$$y' = \omega(\mu)x + \alpha(\mu)y + g(x,y,\mu)$$

change to polar coordinates:

$$x(t) = r(t)\cos\theta(t)$$
$$y(t) = r(t)\sin\theta(t),$$ $r(t), \theta(t)$.. new unknown functions

$\leadsto$
$$r' = \alpha(\mu)r + R(r,\theta,\mu)$$
$$\theta' = \omega(\mu) + Q(r,\theta,\mu)$$

where:
$$R(r,\theta,\mu) = f(r\cos\theta, r\sin\theta)\cdot\cos\theta + g(r\cos\theta, r\sin\theta)\cdot\sin\theta$$
$$rQ(r,\theta,\mu) = -f(\ldots)\sin\theta + g(\ldots)\cos\theta$$

assumptions on $f, g \Rightarrow |f|, |g| = \mathcal{O}(x^2+y^2)$

hence $|R| = \mathcal{O}(r^2)$

$|Q| = \mathcal{O}(r)$.

TRICK: close to 0: $|\omega(\mu)| \geq \frac{1}{2}|\omega(0)| \neq 0$

μ, r $\qquad |Q| \leq \frac{1}{4}|\omega(0)|$

$\Rightarrow |\theta'| \geq \frac{1}{4}|\omega(0)|$; $t \longrightarrow \theta(t)$ 1-1

we can express $r = r(\theta)$ ← new independent variable.

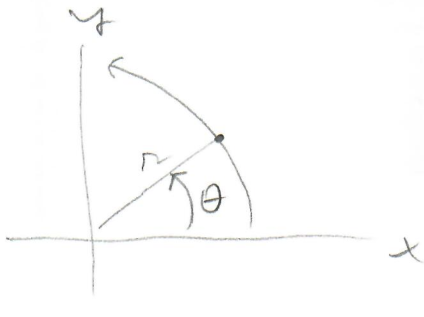
$$\frac{dr}{d\theta} = \frac{\frac{dr}{dt}}{\frac{d\theta}{dt}} = \frac{r'}{\theta'} = \frac{\alpha(\mu)r + R(r,\theta,\mu)}{\omega(\mu) + Q(r,\theta,\mu)}$$

$$= \lambda(\mu) r + P(r,\theta,\mu) \qquad (\widetilde{19.3})$$

where $\lambda(\mu) = \dfrac{\alpha(\mu)}{\omega(\mu)}$, $|P| = \mathcal{O}(r^2)$

aside: $\dfrac{1}{\omega+Q} = \omega^{-1}(1+\omega^{-1}Q)^{-1} = \omega^{-1}(1+\mathcal{O}(Q))$

hence
$$\frac{\alpha r + R}{\omega + Q} = (\alpha r + R)\cdot\frac{1}{\omega}(1+\mathcal{O}(r))$$
$$= \underbrace{\frac{\alpha}{\omega}}_{\lambda} r + \underbrace{\frac{1}{\omega}\Big(\alpha r\,\mathcal{O}(r) + R(1+\mathcal{O}(r))\Big)}_{P = \mathcal{O}(r^2)}$$



we know:
$$\left.\begin{aligned} P(0,\theta,\mu)&=0\\ \partial_r P(0,\theta,\mu)&=0 \end{aligned}\right\} \Leftarrow |P| = \mathcal{O}(r^2)$$

(w.l.o.g.)
$$\lambda(0) = \frac{\alpha(0)}{\omega(0)} = 0$$
$$\lambda'(0) = \frac{\alpha'(0)}{\omega(0)} \neq 0 .$$

key observation: $x(t), y(t)$ per. solution to (19.3) (any period) $\Leftrightarrow$ $r(\theta)$ is 2π-per. solution to $\widetilde{(19.3)}$

denote $\hat{r}(\theta) = \hat{r}(\theta, a, \mu)$ .. solution operator to $\widetilde{(19.3)}$, $r(0) = a$

$$\hat{r}' = \frac{d\hat{r}}{d\theta}$$

$\widetilde{(19.3)}$: $\hat{r}'(\theta) - \lambda(\mu)\hat{r}(\theta) = P(\theta, \hat{r}(\theta), \mu) \quad \Big/ \, e^{-\lambda(\mu)\theta}$

$$\int_0^{2\pi} \quad \frac{d}{d\theta}\, \hat{r}(\theta) e^{-\lambda(\mu)\theta} = e^{-\lambda(\mu)\theta} P(\theta, \hat{r}(\theta), \mu)$$

$$\hat{r}(2\pi) e^{-2\pi\lambda(\mu)} - \underbrace{\hat{r}(0)}_{=a} = \int_0^{2\pi} e^{-\lambda(\mu)\theta} P(\theta, \hat{r}(\theta), \mu)\, d\theta$$

2π-per. solution $\Leftrightarrow$ $\hat{r}(2\pi) = a$; i.e. we obtain

$$0 = a\left(1 - e^{-2\pi\lambda(\mu)}\right) + \int_0^{2\pi} e^{-\lambda(\mu)\theta} P(\theta, \hat{r}(\theta, a, \mu), \mu)\, d\theta$$

"bifurcation equation"

$0 = h(a,\mu)$ ... strategy: division lemma & IFT.

$h(0,\mu) = 0$ since: $a = \hat{r}(0) = 0 \Rightarrow \hat{r}(\theta) \equiv 0$,

i.e. $\hat{r}(\theta, 0, \mu) = 0 \;\forall \theta, \mu$

and $P(\theta, 0, \mu) = 0$.

L.19.1 $\Rightarrow$ $h(a,\mu) = a\,H(a,\mu)$ close to $(0,0)$

where $H(0,0) = \partial_a h(0,0)$

$\partial_\mu H(0,0) = \partial^2_{a\mu} h(0,0)$ by formulas of L.19.1.

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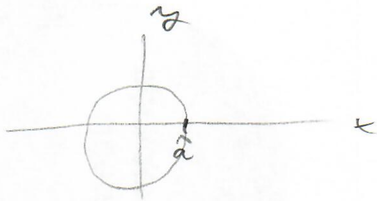
we will show (*) $\partial_a h(0,0) = 0$

$\partial^2_{a\mu} h(0,0) \neq 0$



then by IFT: $H(a,\mu) = 0$ close to $(0,0)$ iff

$\mu = \hat{\mu}(a)$, $\hat{\mu}(\cdot)$ smooth

periodic solution with $r(0) = a$, i.e. non-trivial if $a \neq 0$. $\hat{\mu}(0) = 0$.

$x(0) = a$

$y(0) = 0$

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towards (*):

$$\partial_a h(a,\mu) = 1 - e^{-2\pi\lambda(\mu)} + \int_0^{2\pi} e^{-\lambda(\mu)\theta}\, \partial_r P(\theta, \hat{r}(\theta,a,\mu),\mu) \cdot \partial_a \hat{r}(\theta,a,\mu)\, d\theta$$

$(a,\mu) = (0,0)$: $\lambda(0) = 0$

$\hat{r} = 0$, $\partial_r P = 0$ $\Rightarrow$ $\partial_a h(0,0) = 0$.

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further: $\partial_a h(0,\mu) = 1 - e^{-2\pi\lambda(\mu)}$, since $a = 0 \Rightarrow \hat{r} \equiv 0$, $\partial_r P(\cdots) \equiv 0$

$\partial^2_{a\mu} h(0,0) = 2\pi \cdot \lambda'(0)\, e^{-2\pi\lambda(0)} \neq 0$.

Def. [Bifurcation – abstract.]

Consider $F(u,\lambda): X \times \mathbb{R} \to Y$; X, Y .. Banach

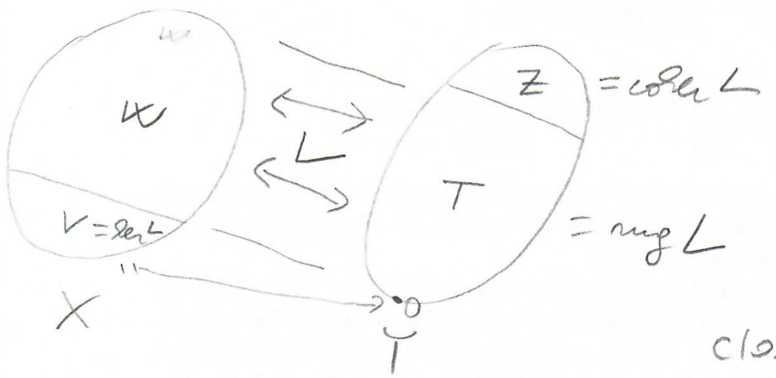
$$F(0,\lambda) = 0 \quad \forall \lambda.$$

$(0,\lambda_0)$ .. point of bifurc. :: $\exists$ non-trivial ($u \neq 0$) solutions in $\forall$ neighborhood.

Rem. necessary cond: $L = D_u F(0,\lambda_0)$ not isomorph. (abstract IFT).

typically: L Fredholm operator, i.e.

- $\dim \ker L < \infty$
- $\operatorname{rng} L$ closed in Y
- $\dim \operatorname{coker} L < \infty$, $\operatorname{coker} L$ (co-kernel) = complement of $\operatorname{rng} L$ in Y



closed

we can write: $X = V \oplus W$, $Y = T \oplus Z$.

fin. dim.

$\exists$ bdd. projections (not unique in general)

Def. $Q: X \to M$ is projection :: is onto, $Q^2 = Q$

Theorem 19.6 [Bifurcation of a simple eigenvalue.]

Assume: $\lambda_0 \in \mathbb{R}$ be s.t. $L = D_u F(0, \lambda_0)$ is Fredholm with $\dim V = \dim Z = 1$ (see above notation)

Let $\exists\, Q : Y \to Z$ projection s.t.

$$Q D_{u\lambda} F(0, \lambda_0)[\phi] \neq 0, \text{ where}$$

$$V = \ker L = \operatorname{Span}\{\phi\},\ \phi \in X.$$

Then: $\exists$ a smooth curve $(u(\alpha), \lambda(\alpha)) : \mathcal{U}(0) \to X \times \mathbb{R}$ solutions to (19.6) s.t. $\lambda(0) = \lambda_0$, $u(0) = 0$ but $u(\alpha) \neq 0\ \forall \alpha \neq 0$.

Pf: consider $F(u, \lambda) = 0$ .. $u = \alpha\phi + w$; $\alpha \in \mathbb{R}$, $w \in W$

$$F = QF + (I - Q)F.$$

$$QF(\alpha\phi + w, \lambda) = 0$$

$$(I - Q)F(\alpha\phi + w, \lambda) = 0 \qquad (\alpha, w, \lambda) \in \mathbb{R} \times W \times \mathbb{R}$$

observe: 2^{nd} eq $\iff$ $w = \tilde{w}(\alpha, \lambda)$, where

$$\tilde{w}(0, \lambda) = 0$$
$$\partial_\alpha \tilde{w}(0, \lambda) = 0$$

by IFT.

so-called Lyapunov Schmidt reduction

$$D_w (I - Q) F(\alpha\phi + w, \lambda)\Big|_{(0,0,0)} = (I - Q)L\big|_W \quad \text{-- isomorphism}$$

$$(I - Q)F(\alpha\phi + \tilde{w}(\alpha, \lambda), \lambda) \equiv 0 \qquad \frac{\partial}{\partial\alpha}\Big|_{(0,0,0)}$$

$$(I - Q)L\phi + (I - Q)L\, \partial_\alpha \tilde{w}(0, \lambda) = 0 \implies \partial_\alpha \tilde{w}(0, \lambda) = 0$$

hence: $F(u,\lambda)=0 \iff \underbrace{QF(\alpha\phi+\tilde{w}(\alpha,\lambda),\lambda)=0}_{g(\alpha,\lambda)\,:\,\mathbb{R}^2\to\mathbb{R}}$

"bifurcation equation".

strategy: division lemma & IFT. at $(u,\lambda)=(0,\lambda_0)$

$$g(0,\lambda)=0 \Rightarrow g(\alpha,\lambda)=\alpha\, G(\alpha,\lambda).$$

$$\partial_\alpha g(\alpha,\lambda)=QD_uF(\alpha\phi+\tilde{w}(\alpha,\lambda),\lambda)[\phi+\partial_\alpha\tilde{w}(\alpha,\lambda)]$$

$$\partial_\alpha g(0,\lambda_0)=QL[\underset{\ker L}{\phi}+\underbrace{\partial_\alpha\tilde{w}(0,\lambda_0)}_{=0}]=0 \Rightarrow G(0,\lambda_0)=0$$

$$\partial_\lambda G(0,0)=\partial^2_{\alpha\lambda}g(0,0)=QD_{u\lambda}F(0,\lambda_0)[\phi]\neq 0$$

by a similar comput. by assumptions

IFT: $G(\alpha,\lambda)=0 \iff \lambda=\lambda(\alpha)$ smooth $\lambda(0)=\lambda_0$

$\to$ solution of $F(u,\lambda)$ in the form

$$u=\alpha\phi+\tilde{w}(\alpha,\lambda)$$
$$\lambda=\lambda(\alpha)$$

but: $\tilde{w}(0,\lambda)=0,\ \partial_\alpha\tilde{w}(0,\lambda)=0$.

$\Rightarrow u=\alpha\phi+\mathcal{O}(\alpha^2)$; $\phi\neq 0$

$\therefore$ non-trivial for α small enough.