20. (Theory of) invariant manifolds

Problem. (1) $x' = Ax + f(x,y)$ $\quad x \in \mathbb{R}^n$ ("center-unstable")

$y' = By + g(x,y)$ $\quad y \in \mathbb{R}^m$ ("stable")

assume. $\operatorname{Re}\sigma(A) \geq 0$ $\quad$ ($\Leftrightarrow$ $x \cdot Ax \geq -\varepsilon |x|^2$, $\varepsilon > 0$ small)

$\operatorname{Re}\sigma(B) < 0$ $\quad$ ($\Leftrightarrow$ $\|e^{tB}\| \leq c_0 e^{-t\beta}$, $\forall t \geq 0$)

$c_0, \beta > 0$

$f, g = 0$ at $(x,y) = (0,0)$

$|f|, |g| \leq \rho$, $\operatorname{Lip} f, g \leq \sigma$ in $\mathbb{R}^{n+m}$

Goal. $\exists\, \phi \in \mathcal{X}$ invariant manifold (center-unstable)

where $\mathcal{X} = \{\phi : \mathbb{R}^n \to \mathbb{R}^m\ ;\ \phi(0) = 0, |\phi| \leq b, \operatorname{Lip}\phi \leq l\}$

satisfying $\boxed{\text{(INV)}: (x(t), y(t)) \text{ solves } (1),\ y(0) = \phi(x(0)) \Rightarrow y(t) = \phi(x(t))\ \forall t \in \mathbb{R}}$

i.e. $\operatorname{Gr}\phi = \{(x,y) \in \mathbb{R}^{n+m},\ y = \phi(x)\}$ is invariant w.r.t. (1).

Lemma 20.1 $\phi \in \mathcal{X}$ satisfies (INV) $\Leftrightarrow$ ϕ satisfies (RED),

where $\boxed{\text{(RED)}: p(t) \text{ solves } (2)\ p' = Ap + f(p, \phi(p)) \text{ ("reduced equation")} \Rightarrow (x(t), y(t)) := (p(t), \phi(p(t))) \text{ solves } (1).}$

Pf. "$\Rightarrow$" assume $p(t)$ solves (2). denote $(\tilde{x}(t), \tilde{y}(t))$ solution of (1) with initial cond. $\tilde{x}(0) = p(0)$, $\tilde{y}(0) = \phi(p(0))$

by (INV) : $\tilde{y}(t) = \phi(\tilde{x}(t))\ \forall t \in \mathbb{R}$,

in particular $(1)_1$: $\tilde{x}' = A\tilde{x} + f(\tilde{x}, \phi(\tilde{x}))$

i.e. $\tilde{x}(t)$ solves (2), $\tilde{x}(0) = p(0)$.

uniqueness for (2): $\tilde{x}(t) = p(t)\ \forall t$; i.e. $\tilde{y}(t) = \phi(p(t))$

hence $(p(t), \phi(p(t)))$ solves (1) .. (RED) holds.

"$\Leftarrow$" assume $(x(t), y(t))$ solve (1), $y(0) = \phi(x(0))$.

denote $\tilde{p}(t)$ solution of (2), with i.c. $\tilde{p}(0) = x(0)$.

by (RED) ... $(\tilde{p}(t), \phi(\tilde{p}(t)))$ solves (1), with i.c.

$$(\tilde{p}(0), \phi(\tilde{p}(0))) = (x(0), \underbrace{\phi(x(0))}_{y(0)}).$$

uniqueness for (1): $\tilde{p}(t) = x(t)$

$\phi(\tilde{p}(t)) = y(t)\ \forall t$

hence $y(t) = \phi(x(t))\ \forall t$ ... (INV) holds.

<u>Lemma 20.2.</u> $B \in \mathbb{R}^{m \times m}$, $\operatorname{Re}\sigma(B) < 0$, $\gamma(t)$ cont., bdd on $(-\infty, 0]$. Then $\exists!$ solution of $y' = By + \gamma(t)$, bdd. on $(-\infty, 0]$.

<u>Pf.</u> using v.c. $y(t) = e^{tB} y(0) + \int_0^t e^{(t-s)B} \gamma(s)\, ds \quad \forall t$

$$\Leftrightarrow e^{-tB} y(t) = y(0) + \int_0^t e^{-sB} \gamma(s)\, ds.$$

1. assume $y(t)$ ... solution with $|y(t)| \le c$, $t \in (-\infty, 0]$.

$|e^{-tB} y(t)| \le \|e^{-tB}\| \cdot |y(t)| \le C_0 e^{\beta t} c \to 0$, $t \to -\infty$

$|e^{-sB} \gamma(s)| \le C_0 e^{\beta s} K \in L^1(-\infty, 0)$.

$\Rightarrow y(0) = -\int_{-\infty}^0 e^{-sB} \gamma(s)\, ds$ ... $y(0)$ and hence $y(t)$ uniquely determined.

2. existence: set $y(0) = \int_{-\infty}^0 e^{-sB} \gamma(s)\, ds$; then by (v.c.)

$$y(t) = e^{tB} \left\{ \int_{-\infty}^0 e^{-sB} \gamma(s)\, ds \right\} + \int_0^t e^{(t-s)B} \gamma(s)\, ds =$$

$$= \int_{-\infty}^{t} e^{(t-s)B} \gamma(s)\,ds \; ; \text{ hence } |y(t)| \le \int_{-\infty}^{t} |\;|\,ds \le$$

$$\le \int_{-\infty}^{t} \|e^{(t-s)B}\| \cdot |\gamma(s)|\,ds \le C_0 \cdot K \cdot \int_{-\infty}^{t} e^{-\beta(t-s)}\,ds = \frac{C_0 K}{\beta}.$$

i.e. $y(t)$ bdd on $(-\infty, 0]$. $\forall t \le \beta$

Lemma 20.3 $\phi \in \mathcal{H}$ satisfies (INV) $\Leftrightarrow \phi$ satisfies (FP)

where $\boxed{(FP): \; \phi(p_0) = \int_{-\infty}^{0} e^{-sB} g(p(s), \phi(p(s)))\,ds \quad \forall p_0 \in \mathbb{R}^n \text{ where } p(t) \text{ is a solution of (2) on the (RHS) with i.c. } p(0) = p_0.}$

Pf. enough to show (RED) $\Leftrightarrow$ (FP), since (RED) $\Leftrightarrow$ (INV) by L.20.1.

"$\Rightarrow$": fix $p_0 \in \mathbb{R}^n$ arbitrary

let $p(t)$ solves (2), with i.c. $p(0) = p_0$

using (RED) ... $(p(t), \phi(p(t)))$ solves (1), in particular

$y(t) := \phi(p(t))$ solves $y' = By + \gamma(t)$,

where $\gamma(t) = g(p(t), \phi(p(t)))$.

observe: $y(t), \gamma(t)$ .. bdd., cont. on $(-\infty, 0]$

(ϕ, g .. bdd, lipschitz).

Lemma 20.2: $y(0) = \int_{-\infty}^{0} e^{-sB} \gamma(s)\,ds$

$$\phi(\underbrace{p(0)}_{p_0}) = \int_{-\infty}^{0} e^{-sB} g(p(s), \phi(p(s)))\,ds$$

.. i.e. (FP) holds.

"$\Leftarrow$": $p(t)$ solves (2) $\overset{?}{\longrightarrow}$ $(p(t), \phi(p(t)))$ solve (1)

observe:

(FP) $\Rightarrow$ (GFP): $\quad \phi(p(t_1)) = \int_{0}^{\infty} e^{-sB} g(p(t_1+s), \phi(p(t_1+s)))\,ds$

for $\forall\, p(t)$ solution of (2).
$\forall t_1 \in \mathbb{R}$.

proof of observation: denote $p_1(t) = p(t+t_1)$.

t_1, $p(t)$ given: clearly $p_1(t)$ solves (2),

$p_1(0) = p(t_1) = p_0$

by (FP): $\phi(p_1(0)) = \int_{-\infty}^{0} e^{-sB} g(p_1(s), \phi(p_1(s)))\,ds$

(GFP) for $t_1 = 0$: (FP). $\qquad$ q.e.d.

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denote $\tilde{y}(t)$ .. solution of $y' = By + \gamma(t)$, where

$y(0) = \phi(p(0))$, $\gamma(t) = g(p(t), \phi(p(t)))$.

$$\tilde{y}(0) = \phi(p(0)) \underset{\text{by (FP)}}{=} \int_{-\infty}^{0} e^{-sB} \underbrace{g(p(s), \phi(p(s)))}_{\gamma(s)}\,ds$$

by Lemma 20.2. $\tilde{y}(t)$ bdd on $(-\infty, 0]$.

$t_1 \in \mathbb{R}$ arbitrary; $\tilde{y}_1(t) = \tilde{y}(t_1+t)$ – bdd on $(-\infty, 0]$.

solves $y' = By + \gamma(t_1+t)$

again by Lemma 20.2: $\tilde{y}_1(0) = \int_{-\infty}^{0} e^{-sB} \gamma(t_1+s)\,ds$.

but: LHS: $\tilde{y}_1(0) = \tilde{y}(t_1)$

RHS:

$$\int_{-\infty}^{0} e^{-sB} g(p(t_1+s), \phi(p(t_1+s)))\,ds \overset{\text{(GFP)}}{=} \phi(p(t_1))$$

hence: $\tilde{y}(t) = \phi(p(t)) \quad \forall t \in \mathbb{R}$

i.e. $p' = Ap + f(p, \phi(p))$ ... $(p(t), \phi(p(t)))$

$\tilde{y}' = B\tilde{y} + g(p, \underbrace{\phi(p)}_{=\tilde{y}})$ solves (7) .. (RED) holds.

<u>Theorem 20.1</u> Assume that (C1-C3) hold:

$$\frac{c_0 \rho}{\beta} \le b, \quad \frac{c_0 \sigma (1+\ell)}{\beta - \varepsilon - \sigma(1+\ell)} \le \ell, \quad c_0 \sigma \left(\frac{1}{\beta} + \frac{1+\ell}{\beta - \varepsilon - \sigma(2+\ell)} \right) < 1.$$

Then $\exists ! \, \phi \in \mathcal{X}$, satisfying (INV).

<u>Moreover:</u> 1) If $Dg(0,0) = 0$, then $D\phi(0) = 0$

2) If $f, g \in C^k$, then $\phi \in C^k$.

Pf. strategy: Banach contraction theorem, since by L. 20.1, 3

$\phi \in \mathcal{X}$ has (INV) $\Leftrightarrow$ ϕ is a fixed point of operator $\mathcal{T}$

$$[\mathcal{T}\phi](p_0) = \int_{-\infty}^{0} e^{-sB} g(p(s), \underline{\phi}(p(s)))\, ds, \quad \text{where } p(\cdot) \text{ solves}$$

(2) $p' = Ap + f(p, \underline{\phi}(p))$

$p(0) = p_0$

<u>STEP 0.</u> $\mathcal{X}$ .. complete metric space ÷ closed subset of $C(\mathbb{R}^n, \mathbb{R}^m)$

norm $\|\phi\|_{\mathcal{X}} = \sup_{p_0 \in \mathbb{R}^n} |\phi(p_0)|$.

<u>STEP 1.</u> $\mathcal{T}\mathcal{X} \subset \mathcal{X}$? $\quad p_0 = 0$ ... $p(t) \equiv 0$ from (2), since

$\phi(0) = 0, \; f(0,0) = 0$

$$\mathcal{T}\phi(0) = \int_{-\infty}^{0} e^{-sB} \underbrace{g(0, \phi(0))}_{=0}\, ds = 0$$

$$|J\phi(p_0)| \le \int_{-\infty}^{0} \underbrace{|e^{-sB} g(p(s), \phi(p(s)))|}_{\le C_0 e^{\beta s}\cdot \rho} ds \le C_0\rho \int_{-\infty}^{0} e^{\beta s} ds = \frac{C_0\rho}{\beta}$$

by (C1), $|J\phi(p_0)| \le b$. ... $\mathrm{Lip}\, J\phi \le \ell$?

auxiliary estimates :

(A1) $y' \ge -ay - c \quad \forall t \le 0 \implies y(t) \le e^{-at}\left(y(0) + \frac{c}{a}\right)$

(where $a, c > 0$)

Pf.: integr. factor e^{-at}, $\int_t^0 dt$

(A2) $|\overset{(g)}{f}(p, \phi(p)) - \overset{(g)}{f}(q, \phi(q))| \le \sigma(1+\ell)|p-q|$

Pf: $\pm f(q, \phi(p))$, $\mathrm{Lip}\, f \le \sigma$, $\mathrm{Lip}\, \phi \le \ell$.

(A3) $|\overset{(g)}{f}(p, \phi(p)) - \overset{(g)}{f}(q, \psi(q))| \le \sigma\left((1+\ell)|p-q| + \|\phi - \psi\|_{\mathcal{X}}\right)$

Pf.: $|f(p, \phi(p)) - f(q, \psi(q))| =$

$\le |f(p, \phi(p)) - f(q, \phi(p))| + |f(q, \phi(p)) - f(q, \psi(q))|$

$\le \sigma\left(|p-q| + |\phi(p) - \psi(q)|\right)$

$\le \sigma\left(|p-q| + \underbrace{|\phi(p) - \psi(p)|}_{\le \|\phi-\psi\|_{\mathcal{X}}} + \underbrace{|\psi(p) - \psi(q)|}_{\le \ell|p-q|}\right)$

estimate $\mathrm{Lip}\, J\phi$: fix $p_0, q_0 \in \mathbb{R}^m$, $\phi \in \mathcal{X}$

$$J\phi(p_0) - J\phi(q_0) = \int_{-\infty}^{0} e^{-sB}\{g(p(s), \phi(p(s))) - g(q(s), \phi(q(s)))\}\,ds$$

$p(t)$ -- solves (2) w. i.c. $p(0) = p_0$
$q(t)$ -- " -- " -- $q(0) = q_0$.

estimate $r(t) = p(t) - q(t)$, $t \le 0$.

$$r' = Ar + f(p, \phi(p)) - f(q, \phi(q)) \quad \Big/ \cdot r$$

$$r' \cdot r = \frac{1}{2}\frac{d}{dt}|r|^2 = r \cdot Ar + r \cdot \big(f(p, \phi(p)) - f(q, \phi(q))\big)$$

$$\frac{1}{2}\frac{d}{dt}|r|^2 \ge -\varepsilon|r|^2 - \sigma(1+\ell)|r|^2 \quad \text{by (A2)}$$

$$\Rightarrow \frac{d}{dt}|r|^2 \ge -a|r|^2 ; \quad a = 2(\varepsilon + \sigma(1+\ell))$$

by (A1) ... $|r(t)|^2 \le e^{-at}|r(0)|^2$

$$|r(t)| \le e^{-(\varepsilon+\sigma(1+\ell))t} \cdot |p_0 - q_0| \quad \forall t \le 0.$$

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hence:

$$|J\phi(p_0) - J\phi(q_0)| \le \int_{-\infty}^{0} \|e^{-sB}\| \cdot \underbrace{|g(p(s), \phi(p(s))) - g(q(s), \phi(q(s)))|}_{\le \sigma(1+\ell)|r(s)| \text{ by (A2)}}\,ds$$

$$\le \int_{-\infty}^{0} c_0 e^{\beta s} \cdot \sigma(1+\ell) \cdot e^{-(\varepsilon+\sigma(1+\ell))s} |p_0 - q_0|\,ds$$

$$= c_0 \sigma(1+\ell) \cdot \underbrace{\int_{-\infty}^{0} e^{(\beta - \varepsilon - \sigma(1+\ell))s}\,ds}_{= \frac{1}{\beta - \varepsilon - \sigma(1+\ell)}} \cdot |p_0 - q_0| \le \ell \cdot |p_0 - q_0| \quad \text{by (C2).}$$

STEP 2: $\mathcal{T}: \mathcal{H} \to \mathcal{H}$ .. contraction ?!

fix $p_0 \in \mathbb{R}^m$, $\phi, \psi \in \mathcal{H}$ $\overset{?}{\Longrightarrow}$ $|\mathcal{T}\phi(p_0) - \mathcal{T}\psi(p_0)| \leq \kappa \|\phi - \psi\|_{\mathcal{H}}$

$\kappa < 1$.

$$\mathcal{T}\underline{\underline{\phi}}(p_0) - \mathcal{T}\underset{\approx}{\psi}(p_0) = \int_{-\infty}^{0} e^{-sB}\{g(p(s), \underline{\underline{\phi}}(p(s))) - g(q(s), \underset{\approx}{\psi}(q(s)))\}\, ds$$

$p(t)$.. solves $p' = Ap + f(p, \underline{\underline{\phi}}(p))$, $p(0) = p_0$

$q(t)$.. solves $q' = Aq + f(q, \underset{\approx}{\psi}(q))$, $q(0) = p_0$

estimate $r(t) = p(t) - q(t)$..., for $t \leq 0$:

$$r' = Ar + f(p, \phi(p)) - f(q, \psi(q)) = Ar + F(p, q).$$

by (A3)... $|F(p,q)| \leq \sigma((1+\ell)|r| + \|\phi - \psi\|_{\mathcal{H}})$

$$\frac{d}{dt} \cdot \frac{1}{2}|r|^2 = r' \cdot r = r \cdot Ar + r \cdot F(p,q)$$

$$\geq -\varepsilon|r|^2 - |r| \cdot \sigma((1+\ell)|r| + \|\phi - \psi\|)$$

$$= -(\varepsilon + \sigma(1+\ell))|r|^2 - \sigma \underbrace{|r| \cdot \|\phi - \psi\|}_{\leq |r|^2 + \|\phi - \psi\|^2} \text{; by Young: ineq.}$$

$$\Rightarrow \frac{d}{dt}|r|^2 \geq -2(\varepsilon + \sigma(2+\ell))|r|^2 - 2\sigma\|\phi - \psi\|^2$$

by (A1)... $$|r(t)|^2 \leq \underbrace{\frac{2\sigma}{\varepsilon + \sigma(2+\ell)}}_{\leq 1} \cdot \|\phi - \psi\|^2 \cdot e^{-2(\varepsilon + \sigma(2+\ell))t}$$

hence: $|r(t)| \leq e^{-(\varepsilon + \sigma(2+\ell))t} \cdot \|\phi - \psi\|_{\mathcal{H}}$; $\forall t \leq 0$.

finally: $|J\phi(p_0) - J\psi(p_0)| \le \int_{-\infty}^{0} C_0 e^{\beta s} \cdot \sigma\left((1+\ell)|R(s)| + \|\phi-\psi\|\right) ds$

by (A3), and using estimate for $R(s)$ ..

$\le K \|\phi - \psi\|$; where

$$K = C_0\sigma \int_{-\infty}^{0} e^{\beta s} \cdot \left\{ (1+\ell) \cdot e^{-(\varepsilon + \sigma(2+\ell))s} + 1 \right\} ds$$

$$= C_0\sigma \cdot \left(\frac{1+\ell}{\beta - \varepsilon - \sigma(2+\ell)} + \frac{1}{\beta} \right) < 1 \quad \text{by (C3)}$$

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taking $\sup_{p_0 \in \mathbb{R}^m}$ -- $\|J\phi - J\psi\|_{\mathcal{X}} \le K \|\phi - \psi\|_{\mathcal{X}}$

"Moreover 1": if $\frac{g(x,y)}{|x|+|y|} \to 0$, $(x,y) \to 0$, then $\frac{\phi(p_0)}{p_0} \to 0$, $p_0 \to 0$.

by above: $\phi(p_0) = \int_{-\infty}^{0} e^{-sB} g(p(s), \phi(p(s)))\, ds$, $p(t)$ solves (2) w.i.c. $p(0) = p_0$

we have estimate $|p(t)| \le e^{-\alpha t} |p_0|$, $\alpha = \varepsilon + \sigma(1+\ell) < \beta$.

$$\frac{|\phi(p_0)|}{|p_0|} \le \int_{-\infty}^{0} \underbrace{\|e^{-sB}\| \cdot \frac{|g(p(s), \phi(p(s)))|}{|p_0|}}_{h(p_0,s)} ds$$

we will show $\int_{-\infty}^{0} h(p_0, s)\, ds \to 0$, $p_0 \to 0$

by Lebesgue Theorem

indeed: $$h(p_0,s) \le C_0 e^{\beta s} \underbrace{\frac{|g(p(s),\phi(p(s)))|}{|p(s)|+|\phi(p(s))|}}_{\le \sigma = Lip\, g} \cdot \underbrace{\frac{|p(s)|+|\phi(p(s))|}{|p_0|}}_{\le e^{-\alpha s}}$$

$$\le C_0 \sigma \cdot e^{(\beta-\alpha)s} \in L^1(-\infty,0).$$

(majorant).

$s<0$ fixed: $h(p_0,s) \underset{p_0 \to 0}{\longrightarrow} 0$, by second term,

because: $|p(s)| \le e^{-\alpha s} |p_0| \to 0$

$|\phi(p(s))| \le \ell \cdot |p(s)| \to 0$

and assumption $\quad g(x,y) = o(|x|+|y|)$

$(x,y) \to (0,0)$.

"Moreover 2:" contraction in C^k-norm

.. technical, but similar in principle ...

Def. $\mathcal{K} = \{ X = (x,y) \in \mathbb{R}^{n+m} ;\ |y| \le \mu |x| \}$ "cone" 20.10

$\mathcal{V} = \{ X = (x,y) \in \mathbb{R}^{n+m} ;\ |y| \ge \mu |x| \}$ "shadow"

more generally : $\mathcal{K}(X_0) = \{ X ;\ X - X_0 \in \mathcal{K} \}$

$\mathcal{V}(X_0) = \{ X ;\ X - X_0 \in \mathcal{V} \}$

Lemma 20.4 Let $\mu > 0$ be fixed, $\sigma = \mathrm{Lip}\, f, g$ small. Then

1. positive cone invariance: $X_1(t), X_2(t)$ solve (1), with

$$X_1(0) \in \mathcal{K}(X_2(0)) \Rightarrow X_1(t) \in \mathcal{K}(X_2(t)) ;\ \forall t \ge 0$$

2. exponential stability of shadow: $X_1(t), X_2(t)$ solve (1), with $X_1(t) \in \mathcal{V}(X_2(t))$ $\forall t \in I$ (interval) $\Rightarrow$

$$|X_1(t) - X_2(t)| \le c\, e^{-\gamma(t-s)} |X_1(s) - X_2(s)| \quad \forall t > s \in I$$

with a suitable $c, \gamma > 0$.

Pf. denote $X_1 = (x_1, y_1)$; $\tilde{x} = x_1 - x_2$

$X_2 = (x_2, y_2)$; $\tilde{y} = y_1 - y_2$

(1) $\Rightarrow$ $\tilde{x}' = A\tilde{x} + f(x_1, y_1) - f(x_2, y_2) = A\tilde{x} + \tilde{f}$

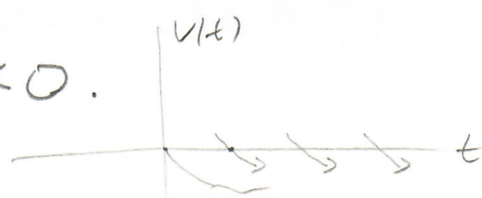
$\tilde{y}' = B\tilde{y} + g(x_1, y_1) - g(x_2, y_2) = B\tilde{y} + \tilde{g}$

we will use repeatedly: $|\tilde{f}|, |\tilde{g}| \le \sigma (|\tilde{x}| + |\tilde{y}|)$

ad 1. set $V(t) = |\tilde{y}(t)|^2 - \mu^2 |\tilde{x}(t)|^2$.

our goal is to show: $V(0) \le 0 \Rightarrow V(t) \le 0$, $\forall t \ge 0$

it will be enough to verify that if $V(t) = 0$, then $V'(t) < 0$.



$$V'(t) = \left(|\tilde{y}(t)|^2 - \mu^2 |\tilde{x}(t)|^2\right)' = 2\tilde{y}\cdot\tilde{y}' - 2\mu^2 \tilde{x}\cdot\tilde{x}'$$

$$= 2\tilde{y}\cdot\left(B\tilde{y} + \tilde{g}\right) - 2\mu^2\tilde{x}\cdot\left(A\tilde{x} + \tilde{f}\right)$$

$$\leq -2\beta|\tilde{y}|^2 + 2|\tilde{y}|\sigma(|\tilde{x}| + |\tilde{y}|) + 2\mu^2\left(\varepsilon|\tilde{x}|^2 + \sigma|\tilde{x}|(|\tilde{x}| + |\tilde{y}|)\right)$$

for $V(t) = 0 \Leftrightarrow |\tilde{y}| = \mu|\tilde{x}|$; hence

$$V'(t) \leq 2|\tilde{x}|^2\left((-\beta+\varepsilon)\mu^2 + \sigma(1 + 2\mu + \mu^2)\right)$$

$$= -c|\tilde{x}|^2; \text{ for } c > 0, \text{ if } \sigma \text{ is small}$$

hence $V'(t) < 0$, unless $\tilde{x}(t) = 0$, which for $V(t) = 0$ means $\tilde{y}(t) = 0$, by uniqueness $\tilde{x}(t) = \tilde{y}(t)\ \forall t$ ... nothing to prove.

<u>ad 2.</u> similarly: let $|\tilde{y}(t)| \geq \mu|\tilde{x}(t)|\ \forall t \in I$, then

$$(|\tilde{y}|^2)' = 2\tilde{y}\cdot\tilde{y}' = 2\tilde{y}\cdot\left(B\tilde{y} + \tilde{g}\right) \leq -2\beta|\tilde{y}|^2 + 2\sigma|\tilde{y}|\cdot(|\tilde{x}| + |\tilde{y}|)$$

$$\leq -2|\tilde{y}|^2\left(\beta - \sigma(1 + \mu^{-1})\right); \text{ since } |\tilde{x}| \leq \mu^{-1}|\tilde{y}|$$

$$= -2\gamma|\tilde{y}|^2; \quad \gamma > 0 \quad (\sigma \text{ small again}).$$

integrate: $|\tilde{y}(t)| \leq e^{-\gamma(t-s)}|\tilde{y}(s)| \quad \forall t > s \in I$

but: RHS : $|\tilde{y}(s)| = |y_1(s) - y_2(s)| \leq |X_1(s) - X_2(s)|$

LHS : $|\tilde{y}(t)| = \dfrac{1+\mu^{-1}}{1+\mu^{-1}}|\tilde{y}(t)| = (1+\mu^{-1})^{-1}\left(|\tilde{y}(t)| + \mu^{-1}|\tilde{y}(t)|\right)$

$$\geq (1+\mu^{-1})^{-1}\underbrace{\left(|\tilde{y}(t)| + |\tilde{x}(t)|\right)}_{= |X_1(t) - X_2(t)|}$$

Theorem 20.2 [Tracking property – asymptotic completeness.]

Let $\phi \in \mathcal{X}$ be as in Theorem 20.1. Let $\mu > \ell$ be fixed, and σ small (as in L. 20.4.). Then for any $X = (x, y)$ a solution to (1) $\exists$ p a solution to (2) s.t. $|X(t) - P(t)| \leq$ $\leq C e^{-\gamma t} |X(0) - P(0)|$; $\forall t \geq 0$, where $P(t) = (p(t), \phi(p(t)))$.

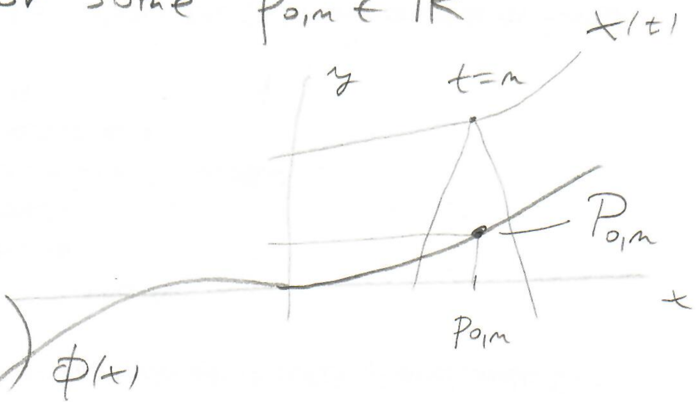
Moreover: $X(0)$ small $\Rightarrow$ $p(0)$ can be taken small.

Pf. for $m = 1, 2, \ldots$ choose $P_{0,m} \in \operatorname{graph} \phi \cap \mathcal{V}(X(m))$

i.e. $P_{0,m} = (p_{0,m}, \phi(p_{0,m}))$ for some $p_{0,m} \in \mathbb{R}^m$

denote $p_m(t)$ .. solution of (2)

s.t. $p(m) = p_{0,m}$



by (INV): $P_m(t) = (p_m(t), \phi(p_m(t)))$

solves (1), $P_m(0) = P_{0,m}$.

key observation: $P_m(t) \in \operatorname{int} \mathcal{V}(X(t))$ $\forall t \leq m$

– by L. 20.4: $\mathcal{K}(X(t))$ pos. invariant $\Longleftrightarrow$

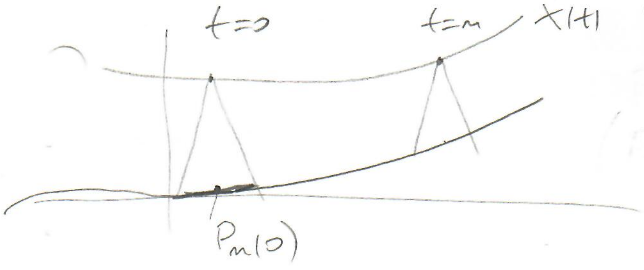
$\operatorname{int} \mathcal{V}(X(t)) = (\mathcal{K}(X(t)))^c$ neg. inv.

furthermore: $M = \mathcal{V}(X(0)) \cap \operatorname{graph} \phi$ is bdd .. obvious

by geometry: $\mu > \ell$
$\phantom{by geometry: \mu >{}}\overset{\shortparallel}{\mathrm{Lip}\,\phi}$

$\Rightarrow$ $\exists$ a subsequence $P_m(0) \to P_0$ in M

$P_0 = (p_0, \phi(p_0))$



by $\mu \to 0$ we easily obtain: $P_\mu(t) \overset{loc}{\rightrightarrows} P(t)$, where

$p(t)$ solves (2), $p(0) = p_0$,

$P(t) = (p(t), \phi(p(t)))$ solves (1) by (INV)

$P(t) \in \mathcal{V}(X(t))$ for all $t \in \mathbb{R}$.

Lemma 20.4: $|X(t) - P(t)| \le c e^{-\gamma t} |X(0) - P(0)|$, $\forall t \ge 0$

... moreover: $X(0) \to 0 \Rightarrow \Pi = \mathcal{V}(X(0)) \cap \operatorname{graph} \phi \to 0$
(again by $\mu \to 0$).

Corollary. [Principle of reduction of stability.]

$(0,0)$ is (asympt.) stable for (1) $\Leftrightarrow$ 0 has this property for (2).

Pf. several implications; obvious are

$(0,0)$ (as.) stable for (1) $\Rightarrow$ 0 (as.) stable for (2)

0 unstable for (2) $\Rightarrow$ $(0,0)$ unstable for (1) ... by (RED),

(2) is a special case of (1).

we show: 0 (as.) stable for (2) $\Rightarrow$ $(0,0)$ (as.) stable for (1).

$X = (x, y)$ solve (1), $X(0)$ close to $(0,0)$.

by Thm. 20.2. $\exists$ $P(0) \in \operatorname{graph} \phi$, close to $(0,0)$, s.t.

$$|X(t) - P(t)| \le c e^{-\gamma t} |X(0) - P(0)|;$$

(2) (as.) stab. $\Rightarrow$ $p(t)$ close to 0 ($\to 0$) for $t \to \infty$

$\Rightarrow$ idem for $P(t) = (p(t), \phi(p(t)))$

$\Rightarrow$ for $X(t)$ by above...

Theorem 20.3 [Approximation of c.m.]

Let: assumptions of Thm 20.1 hold; $\phi(x) \in \mathcal{X}$ be the c.m.

$\psi(x) \in C^1(\mathbb{R}^m, \mathbb{R}^m)$; $\psi(0) = 0$, $D\psi(0) = 0$

$\Pi\psi(x) = \mathcal{O}(|x|^q)$, $|x| \to 0$, for some $q > 1$.

Then: $\phi(x) = \psi(x) + \mathcal{O}(|x|^q)$, $|x| \to 0$.

Pf.: define $\mathcal{Y} = \{\phi(x) \in \mathcal{X} ; |\phi(x)| \leq K|x|^q \; \forall x \in \mathbb{R}^m\}$

$K > 0$ large to be specified later

operator $\mathcal{S} : \phi \longmapsto \mathcal{T}(\phi + \theta) - \theta$; where $\mathcal{T}, \mathcal{X}$ are as in Thm 20.1

$\theta = \theta(x)$ is a C^1 function s.t. $\theta(x) = \psi(x)$ on $\mathcal{U}(0,\delta)$

$= 0$ outside $\mathcal{U}(0,2\delta)$

---

verify : $\boxed{\mathcal{S}\mathcal{Y} \subset \mathcal{Y}}$ by suitable choice of $K, \theta, \ldots$

$\Rightarrow$ we are done : $\mathcal{Y} \subset \mathcal{X}$ is closed ; $\mathcal{S}$ contranction ($\Leftarrow \mathcal{T}$ contraction)

$\Rightarrow \exists! \; \tilde{\phi} \in \mathcal{Y}$ s.t. $\mathcal{S}\tilde{\phi} = \tilde{\phi}$

i.e. $\mathcal{T}(\tilde{\phi} + \theta) = \tilde{\phi} + \theta$

thus $\tilde{\phi} + \theta \in \mathcal{Y} \subset \mathcal{X}$ is a fixed point of $\mathcal{T}$

and so $\tilde{\phi} + \theta = \phi$ ... c.m. from Thm 20.1

$$\phi(x) - \psi(x) \underset{\substack{\uparrow \\ x \text{ small}}}{=} \phi(x) - \theta(x) = \tilde{\phi}(x) = \mathcal{O}(|x|^q)$$

since $\tilde{\phi} \in \mathcal{Y}$

i.e. $\phi(x) = \psi(x) + \mathcal{O}(|x|^q)$, $|x| \to 0$ as required.

... need to show: $\phi \in \mathcal{Y} \Rightarrow \mathcal{S}\phi \in \mathcal{Y}$, i.e. $|\mathcal{S}\phi(p_0)| \le b$

$\mathrm{Lip}\, \mathcal{S}\phi \le \ell$

$|\mathcal{S}\phi(p_0)| \le K |p_0|^q$

$\forall p_0 \in \mathbb{R}^m$

$$\mathcal{S}\phi(p_0) = [\mathcal{T}(\phi+\theta)](p_0) - \theta(p_0) = S_1 + S_2 .$$

$$S_1 = \int_{-\infty}^{0} e^{-sB} g(p(s), \phi(p(s)) + \theta(p(s)))\, ds$$

where $p(\cdot)$ solves $p' = Ap + f(p, \phi(p) + \theta(p))$

$p(0) = p_0$

$$S_2 = -\theta(p_0) \overset{\text{TRICK!!}}{=} -\left[e^{-sB} \theta(p(s)) \right]_{s=-\infty}^{s=0} = -\int_{-\infty}^{0} \frac{d}{ds}[\ldots]\, ds$$

$$= \int_{-\infty}^{0} e^{-sB} \Big\{ B\theta(p(s)) - \underbrace{\frac{d}{ds}\theta(p(s))}_{\substack{\| \\ \nabla\theta(p(s))\, p'(s) =}} \Big\}\, ds$$

$$= \nabla\theta(p(s)) \left[Ap(s) + f(p(s), \phi(p(s)) + \theta(p(s))) \right]$$

---

thus we obtain: $[\mathcal{S}\phi](p_0) = \int_{-\infty}^{0} e^{-sB} Q(p(s))\, ds$

$$Q(p) = B\theta(p) + g(p, \phi(p) + \theta(p)) - \nabla\theta(p)\left[Ap + f(p, \phi(p) + \theta(p)) \right]$$

$$= M\theta(p) + G(p) - \nabla\theta(p) F(p) \quad \text{, where}$$

$$G(p) = g(p, \phi(p) + \theta(p)) - g(p, \theta(p))$$

$$F(p) = f(p, \phi(p) + \theta(p)) - f(p, \theta(p))$$

one shows $|\mathcal{S}\phi(p_0)| \leq b$

$\mathrm{Lip}\, \mathcal{S}\phi(p_0) \leq \ell$ similarly as in Thm 20.1 for operator $\mathcal{T}$

---

we will only show: $|\mathcal{S}\phi(p_0)| \leq K|p_0|^q$, $\forall p_0 \in \mathbb{R}^n$.

estimate $Q(p)$: $|\Pi\theta(p)| \leq C_1|p|^q$ $\forall q \in \mathbb{R}^n$ for $C_1 > 0$ large enough.

since $\theta(\cdot)$ has compact support and $\theta(x) = \psi(x)$ close to 0.

$|G(p)|, |F(p)| \leq \sigma|\phi(p)| \leq \sigma K|p|^q$;

by $\mathrm{Lip}\, f, g \leq \sigma$, $\phi \in \mathcal{Y}$.

$|\nabla\theta| \leq a$, small, globally.

$$\Rightarrow |Q(p)| \leq (C_1 + \sigma(1+a)K)|p|^q.$$

$$|\mathcal{S}\phi(p_0)| \leq \int_{-\infty}^{0} \|e^{-sB}\| \cdot |Q(p(s))|\, ds$$

$$\leq \int_{-\infty}^{0} C_0 e^{\beta s}(C_1 + \sigma(1+a)K)|p(s)|^q\, ds; \quad |p(s)| \leq e^{-as}|p_0| \quad \forall s \leq 0$$

$$\leq C_0(C_1 + \sigma(1+a)K)\int_{-\infty}^{0} e^{(\beta - aq)s}\, ds\, |p_0|^q \quad a = \varepsilon + \sigma(1+\ell) \text{ as in Thm 20.1}$$

$$= C_0(C_1 + \underbrace{\sigma(1+a)}_{<1}K) \cdot \frac{1}{\beta - aq}|p_0|^q \leq K|p_0|^q$$

if $K > 0$ is large