13. Dynamické systémy

Def. Dynamický systém (d.s.) .... (φ,Ω), kde

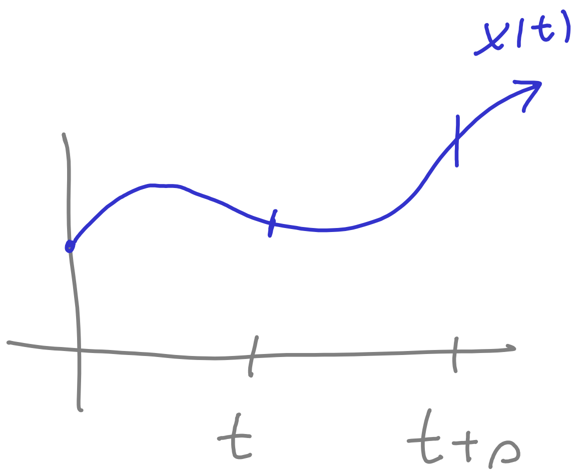
$$\Omega\subset\mathbb{R}^n$$
$$\varphi=\varphi(t,x):\mathbb{R}\times\Omega\to\Omega$$

s.ž.:
(i) $\varphi(0,x)=x \quad \forall x\in\Omega$
(ii) $\varphi(s,\varphi(t,x))=\varphi(s+t,x)$
(iii) $(t,x)\mapsto\varphi(t,x)$ spojité

Příkl.

$$(1)\quad \boxed{\begin{aligned} x'&=f(x)\\ x(0)&=x_0\end{aligned}}$$

$f:\Omega\to\mathbb{R}^n$, $\Omega\subset\mathbb{R}^n$ otevř. lok. lipsch.



$\Downarrow$

řešící fce $\varphi(t,x_0):=\underbrace{x(t)}_{\text{řešení (1)}}$

je d.s.

kanonický příklad

Def. Buď (φ,Ω) d.s. Pak $M\subset\Omega$ nazvu:

- pozitivně invariantní $\iff \varphi(t,x)\in M \quad \forall t\geq 0, x\in M$
- negativně —"— $\iff \varphi(t,x)\in M \quad \forall t\leq 0, x\in M$

- Úplně invariantní $\iff \varphi(t,x) \in M \quad \forall t \in \mathbb{R}, x \in M$

Pro dané $x_0 \in M$ definuji

- pozitivní orbit: $\gamma^+(x_0) = \{\varphi(t,x_0), t \geq 0\}$
- negativní orbit: $\gamma^-(x_0) = \{\varphi(t,x_0), t \leq 0\}$
- úplný orbit: $\gamma(x_0) = \{\varphi(t,x_0), t \in \mathbb{R}\}$

Pozn.: platí: M je inv. (poz./neg.) $\iff \forall x_0 \in M : \gamma_{(+/-)}(x_0) \subset M$

$\gamma_{(+/-)}(x_0)$ je inv. (poz./neg.) d.cv.

Def. Buď (φ, Ω) d.s., $x_0 \in \Omega$. Potom omega-limitní množina bodu x_0 je

$$\omega(x_0) := \{ y \in \Omega, \underbrace{\exists t_k \to \infty \text{ t.ž. } \varphi(t_k, x_0) \underset{k\to\infty}{\to} y}\}$$

Pozn. zobecnění $\lim_{t\to\infty} \varphi(t,x_0)$

ekv. def.: $y \in \omega(x_0) \iff \forall \varepsilon > 0 \, \forall T > 0 \, \exists t > T$ t.ž. $|\varphi(t,x_0) - y| < \varepsilon$

d.cv.

Lemma 13.1 $\omega(x_0) = \bigcap_{\tau>0} \overline{\gamma^+(\varphi(\tau,x_0))}$

Dk. „$\subseteq$" buď $y\in\omega(x_0)$, $\tau>0$ lib.

pozorují: $\gamma^+(\varphi(\tau,x_0)) = \{\underbrace{\varphi(t,\varphi(\tau,x_0))}_{\varphi(t+\tau,x_0)}, t\geq 0\}$

pro $t_k>\tau$ (k velké)

$$\varphi(t_k,x_0) \in \gamma^+(\varphi(\tau,x_0))$$

$$\downarrow$$

$$y\in \overline{\gamma^+(\varphi(\tau,x_0))}$$

„$\supseteq$" buď $y\in$ P.S., tj. $y\in\overline{\gamma^+(\varphi(k,x_0))}$ $\forall k\in\mathbb{N}$

tj. $\exists z_j \in \gamma^+(\varphi(k,x_0))$

t.ž. $z_j\to y$, $j\to\infty$

spec. $\exists z_k\in\gamma^+(\varphi(k,x_0))$

t.ž. $|z_k-y|<\frac{1}{k}$

$$z_k=\varphi(\tau_k,\varphi(k,x_0))$$

$$= \varphi(\underbrace{\tau_k + k}_{\to +\infty}, x_0) \to \underline{y}$$

tj. $y \in \omega(x_0)$

Věta 13.1 [Vlastnosti $\omega(x_0)$].

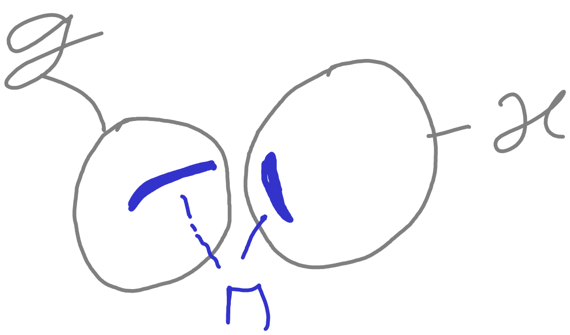
Buď (φ, Ω) d.s., $x_0 \in \Omega$. Potom:

1. $\omega(x_0)$ je uzavřené, (plně) invariantní
2. $\gamma^+(x_0)$ rel. komp. $\Rightarrow$ $\omega(x_0) \neq \emptyset$, kompaktní, souvislé

Pozn.: M není souvislé $\Leftrightarrow$ $\exists\, \mathcal{G}, \mathcal{H}$ otevř. t.ž. $M \subset \mathcal{G} \cup \mathcal{H}$,

$M \cap \mathcal{G} \neq \emptyset$, $M \cap \mathcal{H} \neq \emptyset$

$\mathcal{G} \cap \mathcal{H} = \emptyset$



platí:
- $M \subset \mathbb{R}$ souvislé $\Leftrightarrow$ M je interval
- spojitý obraz souv. mn. je souvislá mn.

Dk. 1. $\omega(x_0)$ inv. $\Leftarrow$ L. 13.1.

? invariance: $\underbrace{y \in \omega(x_0)}$, $t \in \mathbb{R}$ lib.

$\Downarrow$?

$\varphi(t, y) \in \omega(x_0)$

$\exists\, t_k \to \infty$ s.t. $\varphi(t_k, x_0) \to y$

$\underbrace{\varphi(t, \varphi(t_k, x_0))}_{\displaystyle \varphi(t + \underbrace{t_k}_{t'_k \to \infty}, x_0)} \longrightarrow \varphi(t, y)$

(spoj. d.s.)

$\varphi(t + t_k, x_0) \longrightarrow z \in \omega(x_0)$

$\varphi(t, y) = z$

$t + t_k = t'_k \to \infty$

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2. připomenutí: $\gamma^+(x_0)$ rel. komp. $\Leftrightarrow$ $\underbrace{\overline{\gamma^+(x_0)}}_{= K}$ komp.

$\varphi(t_k, x_0) \in K$, $\forall t_k > 0$

$\Rightarrow$ $\exists\, t'_k \to \infty$ s.t. $\varphi(t'_k, x_0)$ konv.

$\Rightarrow$ $\omega(x_0) \neq \emptyset$

$\forall \tau > 0$

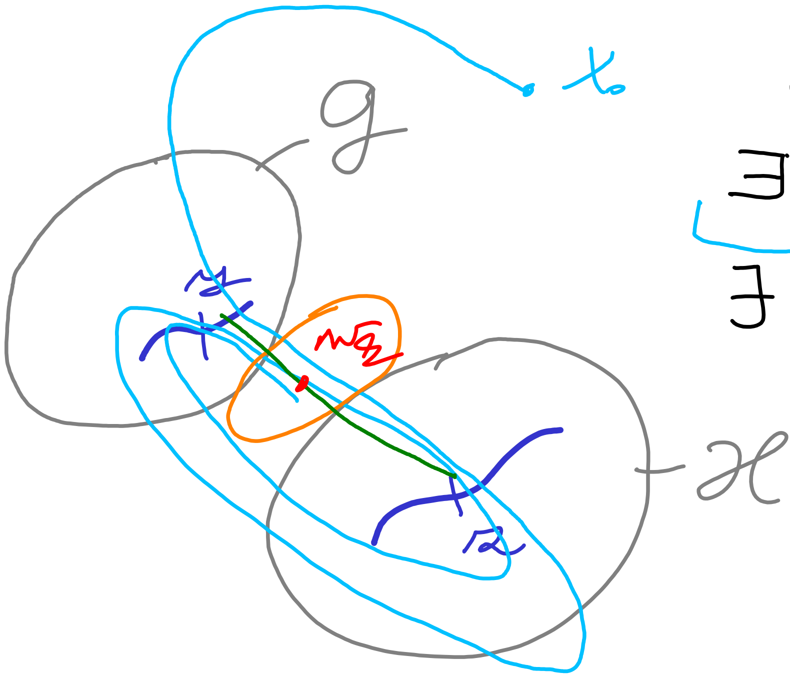
? kompaktní: L. 13.1: $\overline{\gamma^+(\varphi(\tau, x_0))} \subseteq K$

souvislost: ?? $\exists\ \mathcal{G}, \mathcal{H}$ otev., disj.

$\omega(x_0) \subset \mathcal{G} \cup \mathcal{H}$

$\exists\ y \in \omega(x_0) \cap \mathcal{G}$

$\exists\ z \in \omega(x_0) \cap \mathcal{H}$



$\exists\ t_k \to \infty$ t.ž. $\varphi(t_k, x_0) \to y \in \mathcal{G}$

$\exists\ s_k \to \infty$ t.ž. $\varphi(s_k, x_0) \to z \in \mathcal{H}$

BÚNO: $t_k < s_k < t_{k+1} < s_{k+1}$

$\varphi(t_k, x_0) \in \mathcal{G}$, $\varphi(s_k, x_0) \in \mathcal{H}$

polož $O_k = \{\varphi(t, x_0);\ t \in [t_k, s_k]\}$

pozorování: O_k souvislé (spoj. obraz. intervalu)

$\Rightarrow \exists\ w_k \in O_k \setminus (\mathcal{G} \cup \mathcal{H})$

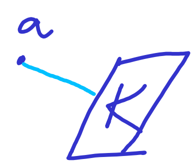
$w_k = \varphi(\tau_k, x_0)$ BÚNO: $w_k \to w \in \omega(x_0)$

(kompaktnost)

avšak $\omega \notin G \cup H$ SPOR

Věta 13.2 Buď (φ, Ω) d.s., $K \subset \Omega$ kompaktní.
Potom $\omega(x_0) = K \iff \operatorname{dist}(\varphi(t, x_0), K) \to 0$
$$t \to \infty$$

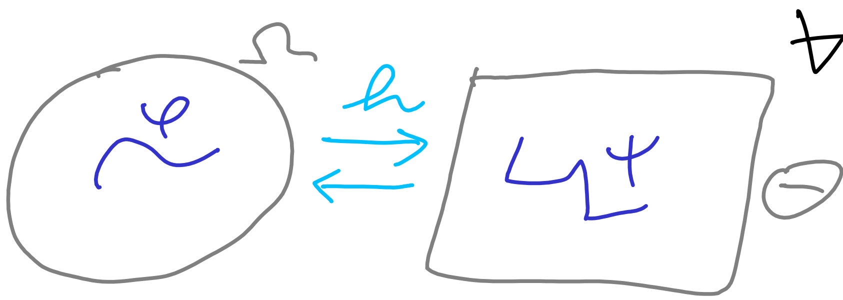
kde $\operatorname{dist}(a, \Gamma) = \inf\{|a - b|, b \in \Gamma\}$



Speciálně: $\omega(x_0) = \{r\} \iff \varphi(t, x_0) \to r,\ t \to \infty$

Dk: d.cv.

Def. Dyn. systémy (φ, Ω) a (ψ, Θ) nazvu topologicky konjugované $\iff \exists$ homeomorfismus $h : \Omega \to \Theta$ t.ž. $h(\varphi(t, x)) = \psi(t, h(x))$
$$\forall t \in \mathbb{R},\ \forall x \in \Omega$$



Ekv.: $\varphi(t,\cdot) = h_{-1} \circ \psi(t,\cdot) \circ h$

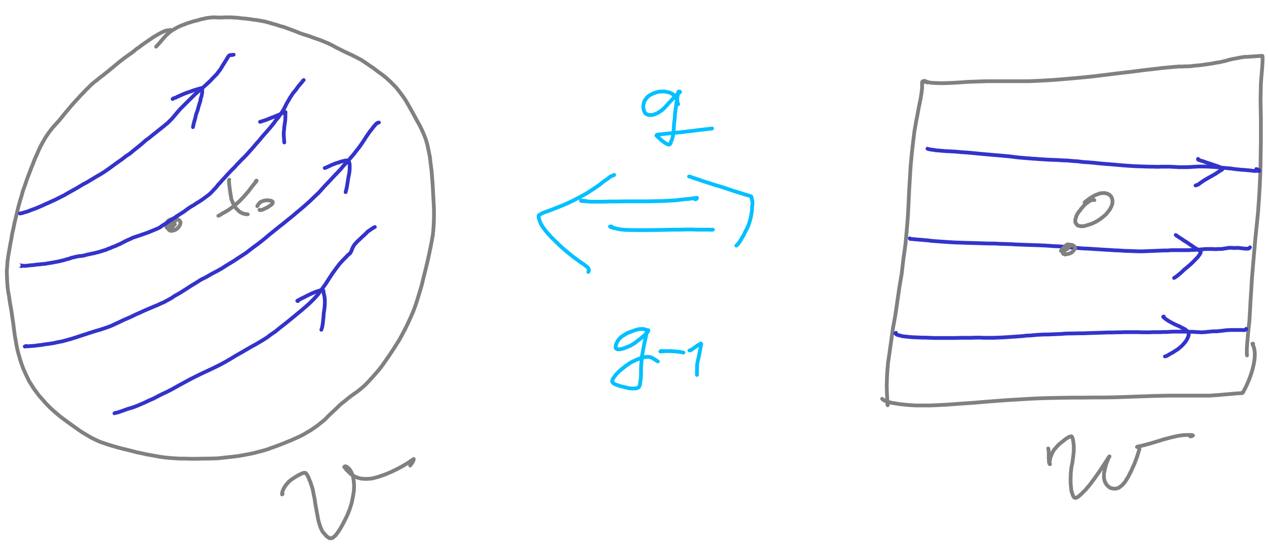
Věta 13.3 [O rektifikaci]

(1) $x' = f(x)$ $\overset{C^r\text{-ekv.}}{\Longleftrightarrow}$ (2) $y' = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

na okolí x_0

kde $f(x_0) \neq 0$

$f \in C^r, r \geq 1$

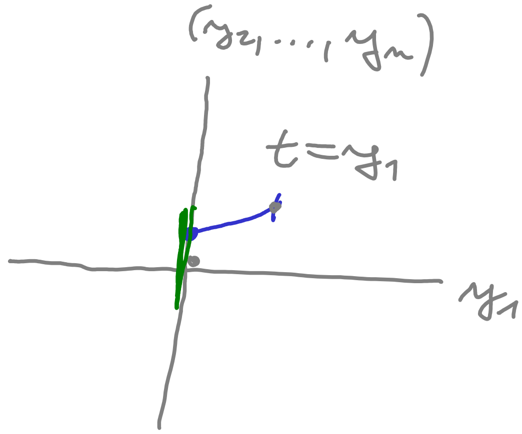


Dk.: KROK 1 BÚNO $x_0 = 0$, $f(x_0) = \begin{pmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

definuj $G : \mathbb{R}^n \to \mathbb{R}^n$

řeš. fce (1)

$(y_1, \dots, y_n) \longmapsto \varphi(y_1, (0, y_2, \dots, y_n))$



W ... dost malé okolí $(x_0=0)$

$\Rightarrow$ G dobře definováno, navíc C^r (ODR, Kap 1–4)

KROK 2: G invertovatelné? $(\Leftarrow \text{VoIF})$

$\nabla G(0) = ?$

- $$\frac{\partial G}{\partial y_1}(0) = \frac{\partial \varphi}{\partial t}(t,y)\Big|_{\substack{t=0\\ y=0}} = f(\underbrace{\varphi(0,0)}_{0}) = \begin{pmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

- $$\frac{\partial G}{\partial y_k}(0) = \frac{\partial}{\partial y_k}\underbrace{\varphi(\boxed{0},(0,y_2,\dots,y_n))}_{=(0,y_2,\dots,y_n)} = \begin{pmatrix} 0 \\ \vdots \\ \boxed{1} \\ 0 \\ \vdots \end{pmatrix}$$

$k=2,\dots,n$ $\qquad$ k-té pozice

$$\Rightarrow \nabla G(0) = \begin{pmatrix} \alpha & ? & \cdots & ? \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix} \Rightarrow \text{regulární!!}$$

$(y_1,\dots,y_n)\in W$ .... dost malé

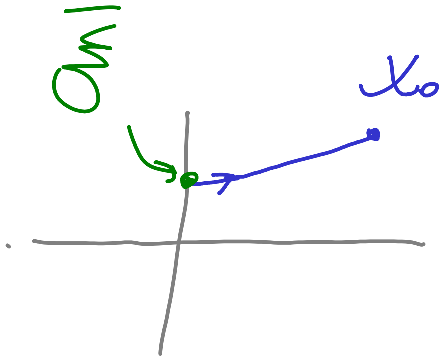
$$\Rightarrow G: W \xrightarrow{1-1} V, \quad G_{-1} \text{ též } C^r$$

okolí $x_0 = 0$.

KROK 3 položme $g := G_{-1} : V \to W$

? lok. ekv., tj. $g(\varphi(t,x_0)) = \psi(t, g(x_0))$ (*)

$\forall x_0 \in V$, $\forall t$ t.ž. $\varphi(t,x_0) \in V$



$$x_0 = \varphi(\tau, \underbrace{(0, \xi_2, \dots, \xi_n)}_{\in M})$$

$$g(x_0) = (\tau, \xi_2, \dots, \xi_n)$$

P.S.: $\psi(t, g(x_0)) = (\tau + t, \xi_2, \dots, \xi_n)$

řešení fce (2)

$$\underbrace{G}_{g^{-1}}(\psi(t, g(x_0))) = \varphi(\tau + t, (0, \xi_2, \dots, \xi_n))$$

$$= \varphi(t, \cancel{g}(x_0))$$

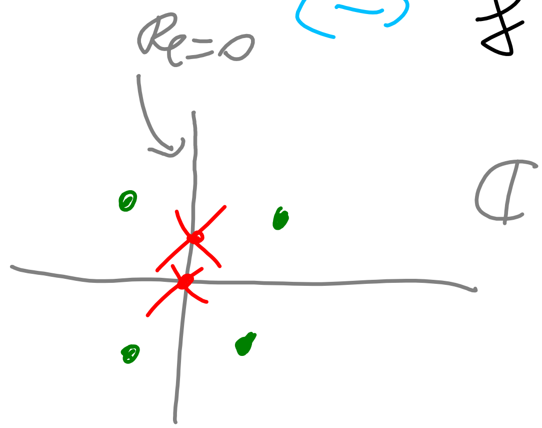
(*) $\varphi(t, x_0) = G(\psi(t, g(x_0)))$

Def: hyperbolický stac. bod (1) $x' = f(x)$

$\iff$ $f(x_0) = 0$, navíc $\forall \lambda \in \sigma(A)$

$\operatorname{Re}\lambda \neq 0$

$A = Df(x_0)$



Věta 13.4. [Hartman–Grobman]

Nechť: x_0 hyp. stac. bod (1), $f \in C^1(\mathcal{U}(x_0))$

Pak: $\exists$ V resp. W okolí x_0 resp. $0 \in \mathbb{R}^n$

t.ž. d.o. určené (1) resp. $\boxed{y' = Ay}$

jsou top. konjugované.