Partial differential equations II

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Introduction

These lecture notes are a transcript of the course "PDE2", given by the second author at MFF UK during spring 2016. All the material presented here is well-known. Precise wording of theorems and their proofs were inspired by various sources we list in the bibliography.
An up-to-date version of these notes, along with supplementary material, can be found at the webpage of the course: `http://www.karlin.mff.cuni.cz/~prazak/vyuka/Pdr2/`.
Feel free to write an e-mail to the authors in the case you would like to share some ideas or report found errors in the text. The authors' email adresses are `radim.cajzl@gmail.com` and `prazak@karlin.mff.cuni.cz`.
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Contents

CHAPTER 1

Vector–valued functions

Vector–valued ... values in infinite–dimensional Banach space.

Notation: we consider:

- $u(t) : I \to X$, $I = [0, T]$ is time interval, X is Banach space,
- $\|u\|_X$ is norm in X
- X^* is dual space X^*, $\langle x^*, x\rangle_{X^*,X}$ is notation of duality
- scalar case: $X = \mathbb{R}$

1.1. Vector–valued integrable functions (Bochner Integral)

Def.: [Simple, weakly and strongly measurable function]. Function $u(t) : I \to X$ is called

simple: if $u(t) = \sum_{j=1}^{N} \chi_{A_j}(t)\, x_j$, where $A_j \subset I$ are (Lebesgue in $\mathbb{R}$) measurable and $x_j \in X$

(strongly) measurable: if there exist simple functions $u_n(t)$ such that $u_n(t) \to u(t)$, $n \to \infty$ for a. e. $t \in I$.

weakly measurable: if the (scalar) function $t \mapsto \langle x^*, u(t)\rangle$ is measurable in I for $\forall x^* \in X^*$.

Remark:

(1) Strongly meas. $\implies$ weakly meas.
(2) $u(t)$ simple $\iff$ $u(t)$ is measurable and $u(I) \subset X$ is finite.

Theorem 1.1 [Pettis characterization of measurability]. Function $u(t) : I \to X$ is measurable iff $u(t)$ is weakly measurable and moreover $\exists N \subset I$ such that $\lambda(N) = 0$ and $u(I \backslash N)$ is separable (i. e., $u(t)$ is "essentially separably–valued").

Corollary.

(1) For X separable: weakly measurable $\iff$ strongly measurable.
(2) $u_n(t)$ is measurable, $u_n(t) \to u(t)$ a. e. $\implies$ $u(t)$ is measurable.
(3) $u(t)$ continuous $\implies$ $u(t)$ measurable

Proof: $u_n(t)$ measurable $\overset{\text{th.1.1}}{\implies}$ $\exists N_n \subset I$, $\lambda(N_n) = 0$. Set $u(I \backslash N_n) = M_n \subset X$ is separable, also $\exists N_0 \subset I$, $\lambda(N_0)$ such that $u_n(t) \to u(t)$ for $\forall t \in I \backslash N_0$. Set $N = \cup_{j=0}^{\infty} N_j$, then $\lambda(N) = 0$ and moreover $u(I \backslash N) \subseteq \overline{\cup_n M^n}^X$ ... separability is preserved under countable unions and closures $\implies$ $u(t)$ is essentially separably–valued.
Is $u(t)$ weakly measurable? Consider $x^* \in X^*$fixed, then the mapping $t \mapsto \langle x^*, u_n(t)\rangle$ is measurable and $\langle x^*, u_n(t)\rangle \to \langle x^*, u(t)\rangle$ as $n \to \infty$ for a. e. $t \in I$, the mapping $t \mapsto \langle x^*, u(t)\rangle$ is measurable by scalar (Lebesgue) theory.
(3) HW1

Def.: [Bochner integral]. function $u(t) : I \to X$ is called Bochner integrable, if there exists $u_n(t)$ simple such that $\int_I \|u(t) - u_n(t)\|\, dt \to 0$, $n \to \infty$. The integral (Bochner integral) is defined as follows

(1) $\int_I u(t)\, dt = \sum_{j=1}^{N} \lambda(A_j)\, x_j$, if $u(t)$ is simple
(2) $\int_I u(t)\, dt = \lim_{n\to\infty} \int_I u_n(t)\, dt$, if $u(t)$ is integrable with $u_n(t)$ simple from the definition above.

Remark: It is possible to show that the definiton is correct, i. e., independent on the choice of A_j, x_j, $u_n(t)$. Also it is possible to prove that $\left\|\int_I u(t)\, dt\right\|_X \le \int \|u(t)\|_X\, dt$.

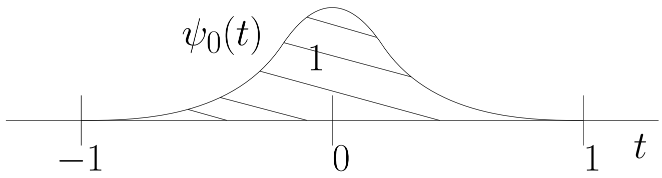

FIGURE 1.1.1. Regularization kernel

Theorem 1.2 [Bochner characterization of measurability]. Function $u(t) : I \to X$ is Bochner integrable iff $u(t)$ is measurable and $\int_I \|u(t)\| \, dt < \infty$.

Theorem 1.3 [Lebesgue]. Let $u_n(t) : I \to X$ be measurable, $u_n(t) \to u(t)$ as $n \to \infty$ for a. e. $t \in I$, let there $\exists g(t) : I \to \mathbb{R}$ integrable and such that $\|u_n(t)\| \le g(t)$ for $\forall n$ and a. e. $t \in I$. Then $\int_I u_n(t)\, dt \to \int_I u(t)\, dt$, moreover $\int_I \|u_n(t) - u(t)\| \, dt \to 0$ for $n \to \infty$ and u is integrable.

Recall: for scalar $u(t) : I \to \mathbb{R}$ we say that $t \in I$ is a Lebesgue point if $\lim_{h\to 0} \frac{1}{2h} \int_{-h}^{h} |u(t+s) - u(t)| \, ds = 0$.
Lebesgue theorem: $u(t) : I \to \mathbb{R}$ is integrable $\implies$ a. e. $t \in I$ is a Lebesgue point.

Def.: [Lebesgue point of function]. We say that $t \in I$ is a Lebesgue point of $u(t) : I \to X$ if

$$\frac{1}{2h} \int_{-h}^{h} \|u(t+s) - u(t)\| \, ds \to 0 \text{ as h}\to 0 .$$

Theorem 1.4 [On Lebesgue points a. e.] If $u(t) : I \to X$ is integrable, then a. e. $t \in I$ is a Lebesgue point.

Remark. Let $u(t) : I \to X$ be integrable.

(1) Set $U(x) = \int_{t_0}^{t} u(t)\, dt$, $t_0 \in I$ fixed $\overset{\text{HW1}}{\Longrightarrow}$ $U'(t) = u(t)$ for every Lebesgue point of $u(t)$, in particular a. e.
(2) (regularization kernels) Let $\psi_0(t) : \mathbb{R} \to \mathbb{R}$ be bounded, zero outside interval $[-1, 1]$ and such that $\int_{-1}^{1} \psi_0(s)\, ds = 1$ (& possibly other regularizations).
$\psi_n(t) = n\psi_0(nt)$, $u * \psi_n(t) = \int_{\mathbb{R}} u(t-s)\, \psi_n(s)\, ds$ for $t \in \left[\frac{1}{n}, T - \frac{1}{n}\right]$ or set $u(t) = 0$ outside I.
Claim: $u * \psi_n(t) \to u(t)$ if $t \in I$ is Lebesgue point of u. Proof:

$$\begin{aligned} u * \psi_n(t) - u(t) &= \int_{\mathbb{R}} u(t-s)\, \psi_n(s)\, ds - u(t) \\ &= \int_{-\frac{1}{n}}^{\frac{1}{n}} [u(t-s) - u(t)]\, \psi_n(s)\, ds \end{aligned}$$

$$\implies \|u\psi_n(t) - u(t)\| \le c \underbrace{n}_{h^{-1}} \underbrace{\int_{-\frac{1}{n}}^{\frac{1}{n}}}_{-h} \|u(t-s)\, u(t)\| \, ds \overset{(*)}{\to} 0, \; n \to \infty$$

where $(*)$ holds by definition of Lebesgue points.

Proof of th. 1.4: By th. 1.1 there $\exists N_0 \subset I$, $\lambda(N_0) = 0$ such that $u(I \backslash N_0) \subseteq \overline{\{x_1, x_2, \dots, x_k, \dots\}}^X$. Set $\varphi_k = \|u(t) - x_k\| \dots$ scalar, integrable. By scalar version of the theorem $\exists N_k \subset I$, $\lambda(N_k) = 0$ such that $\varphi_k(t)$ has Lebesgue point at every $t \in I \backslash N_k$. Set $N = \cup_{k=0}^{\infty} N_k$, then $\lambda(N) = 0$ and $\forall t \in I \backslash N$ is Lebesgue point of $u(t)$, i. e., $\frac{1}{2h} \int_{-h}^{h} \|u(t+s) - u(t)\| \, ds \to 0$, $h \to 0$. Fix $t \in I \backslash N$, $\varepsilon > 0$ arbitrary. Observe: $\exists k$ such that

$\|u(t) - x_k\| \leq \varepsilon$. Then

$$\frac{1}{2h}\int_{-h}^{h}\|u(t+s) - u(t) \pm x_k\|\,ds \quad \leq$$

$$\underbrace{\frac{1}{2h}\int_{-h}^{h}\underbrace{\|u(t+s) - x_k\|}_{\varphi(t+s)}ds}_{I(h)} + \underbrace{\frac{1}{2h}\int_{-h}^{h}\underbrace{\|u(t) - x_k\|}_{\leq\varepsilon}ds}_{\leq\varepsilon\ (*)}$$

$$I(h) = \frac{1}{2h}\int_{-h}^{h}\varphi_k(t+s)\,ds \to \underbrace{\varphi_k(t)}_{<\varepsilon \text{ by choice of } k},\ h \to 0$$

since t is L. point of $\varphi(t)$.
$(*) < 2\varepsilon$ for h small enough.

Def.: [Spaces $L^p(I;X)$]. for $p \in [1,\infty)$ we set

$$\begin{aligned} L^p(I;X) &= \left\{u(t) : I \to X \text{ measurable, } \int_I \|u(t)\|_X^p\,dt < \infty\right\} \\ L^\infty(I;X) &= \Big\{u(t) : I \to X \text{ measurable, essentially bounded,} \\ &\qquad \text{i. e., } \exists c < \infty \text{ s. t. } \|u(t)\|_X < c \text{ a. e.}\Big\} \end{aligned}$$

Remark: These are Banach spaces with the usual norm and the convention that $u(t)$, $\widetilde{u}(t)$ are identified if $u(t) = \widetilde{u}(t)$ a. e. in I. If X is Hilbert with scalar product $(\cdot,\cdot)_X$, then $L^2(I;X)$ is Hilbert with scalar product $(u,v)_{L^2(I;X)} = \int_I (u(t), v(t))_X\,dt$.

Note: $L^1(I;X)$ is space of Bochner integrable functions, $L^p(I;X) \subset L^q(I;X)$ for $p \geq q$ (since I is bounded).

Lemma 1.1 [Approximation and density in $L^p(I;X)$]. Let $p \in [1,\infty)$. Then

(1) Simple functions are dense in $L^p(I;X)$.
(2) Functions of the form $u(t) = \sum_{j=1}^{N}\varphi_j(t)\,x_j$, $\varphi_j(t) \in C_c^\infty(I,\mathbb{R})$ are dense in $L^p(I;X)$
(3) If the space Y is dense in X, then $C_c^\infty(I,Y)$ are dense in $L^p(I;X)$.
(4) Let $\psi_n(t)$ be regularizing kernels, let $u(t) \in L^p(I;X)$ be extended by 0 outside of I. Then $u * \psi_n(t) \to u(t)$ in $L^p(I;X)$ for $n \to \infty$.

Remark:

$$\begin{aligned} C(I,X) &= \{u(t) : I \to X \text{ continuous}\} \\ C^1(I,X) &= C(I,X) \cap \{u'(t) \in C(I,X)\} \\ C_c(I,X) &= C(I,X) \cap \{u(t) = 0 \text{ on } [0,\delta] \cup [T-\delta,T] \text{ for some } \delta\} \end{aligned}$$

Proof:

(1) $u(t) \in L^p(I;X)$ given $\overset{?}{\Longrightarrow} \exists u_n(t)$ simple such that $\int_I \|u_n(t) - u(t)\|_X^p\,dt \to 0$.

We know $u(t)$ is measurable, hence $\exists\widetilde{u}_n(t)$ simple such that $\widetilde{u}_n(t) \to u(t)$ a. e. in I. Set $u_n = \begin{cases}\widetilde{u}_n(t) & \text{if } \|\widetilde{u}_n(t)\| \leq 1 + \|u(t)\| \\ 0 & \text{otherwise}\end{cases}$. Clearly $u_n(t) \to u(t)$, but $\|u_n(t)\| \leq 1 + \|u(t)\|$ for a. e.

$$\begin{aligned} \underbrace{\|u_n(t) - u(t)\|^p}_{\to 0 \text{ s. v.}} &\leq (1 + 2\|u(t)\|)^p \\ &\leq c_p(1 + \|u(t)\|^p) = g(t) \ \ldots \text{ integrable} \end{aligned}$$

By scalar Lebesgue theorem we are done.

(2) by (1) we know: $u(t)=\sum_{j=1}^{N}\chi_{A_j}(t)\,x_j$ are dense in $L^p(I;X)$. Scalar theory: $\chi_{A_j}(t)$ can be approximated in $L^p(I;\mathbb{R})$ by $\varphi_j(t)\in C_c^\infty(I,\mathbb{R})$.
(3) Functions $u(t)=\sum_{j=1}^{N}\varphi_j(t)\,x_j$, $\varphi_j(t)\in C_c^\infty(I,\mathbb{R})$, $x_j\in X$ can be approximated in $L^p(I;X)$ by functions $u(t)=\sum_{j=1}^{N}\underbrace{\varphi_j(t)\,y_j}_{\in C_c^\infty(I;Y),\ \text{obviously}}$, where $\varphi_j\in C_c^\infty(I,\mathbb{R})$, $y_j\in Y$.
(4) $\varepsilon\geq 0$ given, arbitrary. Find $v(t)\in C_c^\infty(I,X)$ such that $\|u(t)-v(t)\|_{L^p(I;X)}<\varepsilon$. Then

$$u*\psi_n(t)\pm v(t)=[u-v]*\psi_n(t)+v*\psi_n(t)=f_n(t)+g_n(t)$$

- $f_n(t)$: use Young inequality: $\|\varphi_1*\varphi_2\|_{L^r}\leq\|\varphi_1\|_{L^p}\|\varphi_2\|_{L^q}$, in particular for $p=r$, $q=1$: $\|\varphi_1*\varphi_2\|_{L^p}\leq\|\varphi_1\|_{L^p}\cdot\|\varphi_2\|_{L^1}$. Set $\varphi_1=u-v$, then $\|u-v\|_{L^p(I;X)}<\varepsilon$, $\varphi_2=\psi_n$, $\|\psi_n\|_{L^1}=1$. $\Longrightarrow$ $\|f_n(t)\|_{L^p(I;X)}<\varepsilon$.
- $g_n(t)=v*\psi_n(t)\rightrightarrows v(t)$ in I, hence $v*\psi(t)\to v(t)$ in $L^p(I;X)$, hence $\|g_n(t)-v(t)\|_{L^p(I;X)}<\varepsilon$ for n large enough.

$$u*\psi_n(t)-u(t)=\underbrace{f_n(t)}_{\to 0}+\underbrace{g_n(t)-v(t)}_{\substack{\|\cdot\|<\varepsilon\\ \text{for } n \text{ large}}}+\underbrace{v(t)-u(t)}_{\|\cdot\|<\varepsilon},$$

therefore $\|u*\psi_n(t)-u(t)\|<3\varepsilon$ for n large.

Remark.
(1) Corollary: X separable $\Longrightarrow L^p(I;X)$ is separable for $p\in[1,\infty)$.
(2) None of these is true for $p=\infty$.
(3) More about $L^p(\ldots)$ can be found below (dual space, geometry).

1.2. *AC* functions and weak time derivative

Def.: $u(t):I\to X$ is absolutely continuous $(AC(I,X))$ $\iff$ for $\forall\varepsilon>0$ $\exists\delta>0$: $\forall$ disjoint finite system $(\alpha_j,\beta_j)\subset I$ it holds that if $\sum_j|\alpha_j-\beta_j|<\delta$ then $\sum_j\|u(\alpha_j)-u(\beta_j)\|<\varepsilon$.

Theorem 1.5 [Derivative of *AC* function]. Let $u(t)\in AC(I,X)$, let X be reflexive and separable. Then $u'(t)=\lim_{h\to 0}\frac{1}{h}(u(t+h)-u(t))$ for a. e. $t\in I$, $u'(t)\in L^1(I;X)$ and $u(t_2)-u(t_1)=\int_{t_1}^{t_2}u'(s)\,ds$ for $\forall t_1,t_2\in I$.

Recall: We assume scalar version ($X=\mathbb{R}$) is already proven.
- $I=[0,T]$
- X is Banach space.

Recall: (Eberlein–Šmulian theorem). X is reflexive, $\|u_n\|$ is bounded $\Longrightarrow$ $\exists$ weakly convergent subsequence $\widetilde{u}_n$, i. e., $\exists u\in X$ such that $\widetilde{u}_n\rightharpoonup u$ meaning $\langle x^*,\widetilde{u}_n\rangle\to\langle x^*,u\rangle$ for $\forall x^*\in X^*$ fixed.

Recall: norm is weakly lower-semicontinuous:

$$u_n\rightharpoonup u\implies\|u\|\leq\liminf_{n\to\infty}\|u_n\|$$

Proof of th. 1.5: **Step 1:** set

$$V(t)=\operatorname{var}(u(t),0,t)\overset{\text{by def.}}{=}\sup_D\sum_{j=1}^{N}\|u(\tau_j)-u(\tau_{j-1})\|$$

where the supremum is taken over all partitions D of the interval $[0,t]$, i. e., $D:0=\tau_0<\tau_1<\ldots<\tau_n=t$. observe: $V(t)\geq 0$, $V(t)$ is non-decreasing, even AC.

$$\left\|\frac{u(t+h)-u(t)}{h}\right\|\leq\frac{V(t+h)-V(t)}{h}\to V'(t)\in\mathbb{R}\text{ for a. e. }t\in I\backslash N_0,$$

where $\lambda(N_0)=0$. Hence $\left\|\frac{u(t+h)-u(t)}{h}\right\|$ is bounded as $h\to 0$, for a. e. $t\in I$.

Step 2: X^* is separable, let $\{x_1^*, x_2^*, \ldots\}$ be some countable dense set in X^*. Define auxiliary functions $\varphi_k(t) = \langle x_k^*, u(t)\rangle : I \to \mathbb{R}$, φ_k are AC. Therefore by scalar case $\varphi'(t) \in \mathbb{R}$, $\exists$ for $\forall t \in I \setminus N_k$, $\lambda(N_k) = 0$ and $\varphi_k(t_2) - \varphi_k(t_1) = \int_{t_1}^{t_2} \varphi'(s)\, ds$ $\forall t_1, t_2$. For $t \in I \setminus \bigcup_{k>0} N_k$ define $v(t) \in X$ as follows: take some $h_n \to 0$ such that $\frac{u(t+h_n)-u(t)}{h_n} \rightharpoonup v(t)$ (by Step 1 & Eberlein–Šmulian).
observe: $v(t)$ is independent of the sequence h_n, hence $\frac{u(t+h)-u(t)}{h} \rightharpoonup v(t)$, $h \to 0$.
Assume: $\frac{u(t+\widetilde{h}_n)-u(t)}{\widetilde{h}_n} \rightharpoonup \widetilde{v}(t) \in X$. But then

$$\begin{array}{ccc} \left\langle x_k^*, \dfrac{u(t+h_n)-u(t)}{h_n} \right\rangle & \to & \langle x_k^*, v(t)\rangle \\ \| & & \| \\ \dfrac{1}{h_n}\left(\varphi_k(t+h_n) - \varphi_k(t)\right) & \to & \varphi_k'(t), \end{array}$$

the same holds with $\widetilde{h}_n$: $\langle x_k^*, \widetilde{v}(t)\rangle = \varphi_k'(t) = \langle x^*, v(t)\rangle$, therefore $\langle x_k^*, \widetilde{v}(t) - v(t)\rangle = 0$ for $\forall k$, by density of x_k^* in X we have $\widetilde{v}(t) = v(t)$.
Step 3: $v(t) \in L^1(I, X)$, i. e.,

(1) measurable ... see excercises, HW1
(2) $\int_I \|v(t)\|\, dt < \infty$ ... weak lower semicontinuity

$$\begin{aligned} \int_I \|v(t)\|\, dt &\le \int_I \liminf_{n\to\infty} \left\| \frac{1}{h_n}\left(v(t+h_n) - v(t)\right) \right\| dt \\ &\le \int_I \liminf_{n\to\infty} \frac{1}{h_n}\left(V(t+h_n) - V(t)\right) dt \\ \text{we know} \quad &= \int_I \lim_{n\to\infty} \frac{1}{h_n}\left(V(t+h_n) - V\right) dt \\ &= \int_I V'(t)\, dt < \infty \end{aligned}$$

since $V(t)$ is AC.
Step 4: $u(t_2) - u(t_1) = \int_{t_1}^{t_2} v(s)\, ds$ for $\forall t_1, t_2 \in I$.
We have:

$$\langle x_k^*, u(t_2)\rangle - \langle x_k^*, u(t_1)\rangle = \int_{t_1}^{t_2} \underbrace{\langle x_k^*, v(s)\rangle}_{\varphi_k'(s)} ds = \left\langle x_k^*, \int_{t_1}^{t_2} v(s)\, ds \right\rangle$$

We conclude by density of x_k^* in X^*.
Step 5: $\frac{u(t+h)-u(t)}{h} \to v(t)$ strongly for a. e. $t \in I$ ... consequence of Th. 1.4 and step 4.

Notation: $D(I) = C_c^\infty(I, \mathbb{R})$ ... scalar test-functions.

Lemma 1.2 [Weak characterization of constant and zero function]. Let $u(t) \in L^1(I; X)$.

(1) If $\int_I u(t)\varphi(t)\, dt = 0$ for $\forall \varphi(t) \in D(I)$ then $u(t) = 0$ a. e. in I.
(2) If $\int_I u(t)\varphi'(t)\, dt = 0$ for $\forall \varphi(t) \in D(I)$ then $\exists x_0 \in X$ such that $u(t) = x_0$ a. e. in I.

Proof: (1): Set $u_n(t) = u * \psi_n(t)$, where $\psi_n(t) = n\psi_0(nt)$, $\psi_0(t)$ is convolution kernel such that $\psi_0 \in D([-1,1])$.
$PS = \int_{\mathbb{R}} u(s) \underbrace{\psi_n(t-s)}_{\in D(I) \text{ for } n \text{ large enough and } t \in [0,T] \text{ fixed}} ds = 0$ by assumptions.
On the other hand: $LS = u_n(t) \to u(t)$ a. e. (by th. 1.4)
(2) $u_n(t)$ as above: $u_n'(t) = \frac{d}{dt}\int_{\mathbb{R}} u(s)\psi_n(t-s)\, ds \overset{\text{ex.}}{=} \int_{\mathbb{R}} u(s) \underbrace{\psi_n'(t-s)}_{-\varphi'(s)} ds = 0$, $t \in (0,T)$ fixed, n large enough,

where $\varphi(s) = \psi_n(t-s) \in D(I)$. Hence $u_n(t)$ is smooth, $u_n'(t) = 0$, therefore $u_n(t) = x_n$ everywhere. But $u_n(t) \to u(t)$ a. e. in I, which concludes the proof.

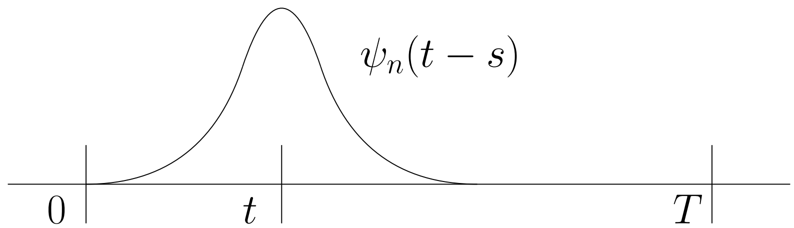

FIGURE 1.2.1. Regularization function $\psi_n(t-s)$

Lemma 1.3 [Equivalent definition of weak derivative]. Let $u(t)$, $g(t) \in L^1(I;X)$. Then the following are equivalent:

(1) $\exists x_0 \in X$ such that $u(t) = x_0 + \int_0^t g(s)\,ds$ for a. e. $t \in I$.
(2) $\int_I u(t)\varphi'(t)\,dt = -\int_I g(t)\varphi(t)\,dt$ for $\forall\varphi(t) \in D(I)$.
(3) $\frac{d}{dt}\langle x^*, u(t)\rangle = \langle x^*, g(t)\rangle$ in the sense of distributions in $(0,T)$ for $\forall x^* \in X^*$ fixed.

Proof:

(2) $\iff$ (3) recall: (3) means $\int_I \langle x^*, u(t)\rangle \varphi'(t)\,dt = -\int_I \langle x^*, g(t)\rangle \varphi(t)\,dt$ for $\forall\varphi \in D(t)$.
It follows that $\left\langle x^*, \int_I u(t)\varphi'(t)\,dt\right\rangle = -\left\langle x^*, \int g(t)\varphi(t)\,dt\right\rangle$ for $\forall x^* \in X^*$ and $\forall\varphi \in D(I)$. Therefore $\int_I u(t)\varphi'(t)\,dt = \int g(t)\varphi(t)\,dt \iff$ (2) holds.

(1) $\implies$ (3) take $x^* \in X^*$ arbitrary, apply to (1):

$$\underbrace{\langle x^*, u(t)\rangle}_{=\xi(t)} = \langle x^*, x_0\rangle + \int_0^t \underbrace{\langle x^*, g(s)\rangle}_{\eta(s)} ds = \widetilde{\xi}(t)$$

clearly $\widetilde{\xi}(t) \in AC$, $\widetilde{\xi}(t) = \xi(t)$ a. e. Take $\varphi(t) \in D(I)$: one easily verifies that $\widetilde{\xi}(t)\varphi(t) \in AC$ and $\left(\widetilde{\xi}(t)\varphi(t)\right)' = \widetilde{\xi}(t)\varphi(t) + \widetilde{\xi}(t)\varphi'(t)$ a. e. Integrate over $[0,T]$, note that $\varphi(0) = \varphi(T) = 0$: $0 = \int_I \underbrace{\widetilde{\xi}(t)}_{=\eta(t)\text{ a. e.}} \varphi(t) + \underbrace{\widetilde{\xi}(t)}_{=\xi(t)\text{ a. e.}} \varphi'(t)\,dt$. Therefore (3) holds.

(2) $\implies$ (1) set $\widetilde{u}(t) = \int_0^t g(s)\,ds \ldots$ $\widetilde{u}(t) \in AC(I,X)$, $\widetilde{u}'(t) = g(t)$ a. e.
Take $\varphi(t) \in D(I)$ arbitrary $\ldots$ $\widetilde{u}(t)\varphi(t) \in AC(I,X)$, by remark below Th. 1.4 it holds that

$$\begin{aligned} (\widetilde{u}(t)\varphi(t))' &= \widetilde{u}'(t)\varphi(t) + \widetilde{u}(t)\varphi'(t) \quad \Big/ \int_I dt \\ \implies \quad 0 &= \int_I g(t)\varphi(t)\,dt + \int_I \widetilde{u}(t)\varphi'(t)\,dt, \end{aligned}$$

by (2) we have: $\int_I g(t)\varphi(t)\,dt + \int_I u(t)\varphi'(t)\,dt = 0$.
Subtract: $\int_I [u(t) - \widetilde{u}(t)]\varphi'(t) = 0 \ \forall\varphi(t) \in D(I) \overset{\text{L. 1.2, 2}}{\implies} \exists x_0 \in X$ s. t. $u(t) - \widetilde{u}(t) = x_0$ a. e. in I.
But $\widetilde{u}(t) = \int_0^t g(s)\,ds$, hence (1) holds.

Def.: [Weak derivative of $u(I) :\to X$]. Let $u(t)$, $g(t) \in L^1(I;X)$. We say that $g(t)$ is weak time derivative of $u(t)$, if one (and hence any) of the assertions of L. 1.3 holds. We write $\frac{d}{dt}u(t) = g(t)$. We then define $W^{1,p}(I;X) = \left\{u(t) \in L^p(I;X), \frac{d}{dt}u(t) \in L^p(I;X)\right\}$.

1.3. Geometry and duality in $L^p(I;X)$ spaces

Recall: [Reflexive space, canonical embedding $X \to X^{**}$]. mapping $J : X \to (X^*)^*$ is isometric into. For X reflexive, J is onto.
$x \mapsto Jx$, where $\langle Jx, y\rangle_{X^{**},X^*} = \langle y, x\rangle_{X^*,X}$ for $\forall y \in X^*$. Eberlein–Šmulian: X reflexive, $\{u_n\} \subset X$ bounded $\implies$ $\exists$ subseq. $\widetilde{u}_n$: $\exists u \in X$ s. t. $\widetilde{u}_n \rightharpoonup u$ (weak convergence, meaning $\langle y, \widetilde{u}_n\rangle \to \langle y, u\rangle$ for $\forall y \in X^*$).

Def.: [Strictly and uniformly convex space]. X is:
strictly convex: $\|x\|, \|y\| \le 1,\ x \neq y \implies \left\|\frac{x+y}{2}\right\| < 1$.
uniformly convex: $\forall \varepsilon > 0\ \exists \delta > 0$ s. t. $\|x\|, \|y\| \le 1,\ \|x-y\| \ge \varepsilon \implies \left\|\frac{x+y}{2}\right\| < 1-\delta$.

Theorem 1.6 [Relationship between weak and strong convergence in UC spaces]. Let X be uniformly convex, let $x_n \rightharpoonup x$ and let $\|x_n\| \to \|x\|$. Then $x_n \to x$.

Proof: WLOG: $x \neq 0$, set $y_n = \frac{x_n}{\|x_n\|}$, $y = \frac{x}{\|x\|}$, clearly $y_n \rightharpoonup y$, $\|y_n\| \to \|y\| = 1$. Assume for contradiction $y_n \not\rightarrow y$, hence $\exists \varepsilon > 0$, $\exists$ subseq $\widetilde{y}_n$ such that $\|\widetilde{y}_n - y\| \ge \varepsilon$ for $\forall n$. But $\widetilde{y}_n \rightharpoonup y = \frac{y+y}{2} \leftharpoondown \frac{y_n+y}{2}$ and since the norm is weakly lower semicontinuous, we have $\|y\| \le \liminf_n \left\|\frac{y_n+y}{2}\right\| \overset{X \text{ unif. conv.}}{\le} 1-\delta < 1$, by which we have obtained a contradiction.

Remark: general principle: $x_n \rightharpoonup x$, $A(x_n) \rightharpoonup A(x)$ for suitable $A \implies x_n \to x$.

Remark:

- X Hilbert $\implies$ X uniformly convex,
- X uniformly convex $\implies$ X reflexive,
- $L^p(\Omega)$ is uniformly convex for $p \in (1, +\infty)$.

Theorem 1.7. Let X be uniformly convex, $p \in (1, +\infty)$. Then $L^p(I;X)$ is uniformly convex.

Recall:

- $p, p' \in [1, \infty)$ are Hölder conjugate $\iff \frac{1}{p} + \frac{1}{p'} = 1$
- Hölder inequality: $\int_\Omega |u(x)\, v(x)|\ dx \le \left(\int_\Omega |u(x)|^p\ dx\right)^{\frac{1}{p}} \left(\int_\Omega |v(x)|^{p'}\right)^{\frac{1}{p'}}$
- consequence: $v(x) \in L^{p'}(\Omega) \ \ldots\ F_v \in (L^p(\Omega))^* : u(\cdot) \mapsto \int_\Omega u(x)\, v(x)\ dx$ for $\forall u \in L^p(\Omega)$.

Theorem 1.8 [Hölder inequality]. Let $u(t) \in L^p(I;X)$, $v(t) \in L^{p'}(I;X^*)$, p, p' Hölder conjugate. Then the mapping $t \mapsto \langle v(t), u(t)\rangle_{X^*,X}$ is measurable and

$$\int_I |\langle v(t), u(t)\rangle|\, dt \le \left(\int_I \|u(t)\|_X^p\right)^{\frac{1}{p}} \left(\int_I \|v(t)\|_{X^*}^{p'}\right)^{\frac{1}{p'}}.$$

Proof: $u(t), v(t)$ are strongly measurable, $PS < \infty$. Then $\exists u_n(t) : I \to X$ simple such that $u_n(t) \to u(t)$ and $\exists v_n(t) : I \to X$ simple such that $v_n(t) \to v(t)$ a. e. in X, X^*. Therefore the mapping $t \mapsto \langle v_n(t), u_n(t)\rangle$ is simple and measurable. Since $\langle v_n(t), u_n(t)\rangle \to \langle v(t), u(t)\rangle$ a. e. ($\langle\cdot,\cdot\rangle$ is continuous), the mapping $t \mapsto \langle v(t), u(t)\rangle$ is measurable. Finally: $|\langle v(t), u(t)\rangle| \le \|v(t)\|_{X^*} \|u(t)\|_X$ and by scalar Hölder inequality we are done.

Corollary: $v(t) \in L^{p'}(I;X^*) \implies F_v : u \mapsto \int_I \langle v(t), u(t)\rangle_{X^*,X}$ is well defined for $\forall u(t) \in L^p(I;X)$.

Theorem 1.9 [Characterization of $L^p(I,X)$ dual]. Let X be reflexive, separable, let $p \in [1, +\infty)$. Denote $\mathscr{X} = L^p(I;X)$. Then for any $F \in \mathscr{X}^*$ there is $v(t) \in L^{p'}(I, X^*)$ such that

$$\langle F, u(t)\rangle_{\mathscr{X}^*,\mathscr{X}} = \int_I \langle v(t), u(t)\rangle_{X^*,X}\ dx \quad \forall u(t) \in \mathscr{X}.$$

Moreover, $v(t)$ is uniquely defined and $\|v(t)\|_{L^{p'}(I;X)} = \|F\|_{\mathscr{X}^*}$.

Proof. **Step 1:** For $\tau \in I$ and $x \in X$ denote $u_{\tau,x}(t) = \chi_{[0,\tau]}(t)x$. Clearly $u_{\tau,x}(t) \in \mathscr{X}$ and $x \mapsto \langle F, u_{\tau,x}(\cdot)\rangle_{\mathscr{X}^*,\mathscr{X}}$ is linear continuous. Hence there is $g(\tau) \in X^*$ such that $\langle F, u_{\tau,x}(\cdot)\rangle_{\mathscr{X}^*,\mathscr{X}} = \langle g(\tau), x\rangle_{X^*,X}$ for all $x \in X$.
We will show there is $v(t) \in L^{p'}(I;X^*)$ such that

$$g(\tau) = \int_0^\tau v(t)\, dt \tag{1}$$

$$\|v(t)\|_{L^{p'}(I;X^*)} \le \|F\|_{\mathscr{X}^*} \tag{2}$$

Observe that with this we are done: (1) implies that

$$\langle F, u_{\tau,x}(\cdot)\rangle_{\mathscr{X}^*,\mathscr{X}} = \left\langle \int_0^\tau v(t)\,dt, x\right\rangle_{X^*,X} = \int_I \langle v(t), u_{\tau,x}(t)\rangle_{X^*,X}\,dt$$

By linearity we have

$$\langle F, u(\cdot)\rangle_{\mathscr{X}^*,\mathscr{X}} = \int_I \langle v(t), u(t)\rangle\,dt \tag{3}$$

for any $u(t) = \sum_j \chi_{(\alpha_j,\beta_j]}(t)x_j$. But such functions are dense in $\mathscr{X}$, to which (3) extends, using continuity of F on the left, and Hölder inequality on the right. Furthermore, it now follows from (3) that

$$\|F\|_{\mathscr{X}^*} = \sup_{\|u(t)\|_{\mathscr{X}}=1} \langle F, u(\cdot)\rangle_{\mathscr{X}^*,\mathscr{X}} = \sup_{\|u(t)\|_{\mathscr{X}}=1} \int_I \langle v(t), u(t)\rangle_{X^*,X}\,dt \le \|v(t)\|_{L^{p'}(I;X^*)}$$

by Hölder inequality again. Together with (2) we obtain $\|F\|_{\mathscr{X}^*} = \|v(t)\|_{L^{p'}(I;X^*)}$; this also implies that $v(t)$ is uniquely defined.

Step 2: Towards proving (1), we first show that $g(t): I \to X^*$ is absolutely continuous. Let $(\alpha_j,\beta_j) \subset I$ be disjoint. It follows from reflexivity of X that there exist $x_j \in X$ with $\|x_j\| = 1$ such that $\|g(\beta_j) - g(\alpha_j)\|_{X^*} \langle g(\beta_j) - g(\alpha_j), x_j\rangle_{X^*,X}$. On the other hand

$$\langle g(\beta_j) - g(\alpha_j), x_j\rangle_{X^*,X} = \left\langle F, u_{\beta_j,x_j}(\cdot) - u_{\alpha_j,x_j}(\cdot)\right\rangle_{\mathscr{X}^*,\mathscr{X}} = \left\langle F, \chi_{(\alpha_j,\beta_j]}(\cdot)x_j\right\rangle_{\mathscr{X}^*,\mathscr{X}}$$

Hence

$$\begin{aligned}\sum_j \|g(\beta_j) - g(\alpha_j)\|_{X^*} &= \left\langle F, \sum_j \chi_{(\alpha_j,\beta_j]}(\cdot)x_j\right\rangle_{\mathscr{X}^*,\mathscr{X}} \\ &\le F_{\mathscr{X}^*}\left\|\sum_j \chi_{(\alpha_j,\beta_j]}(t)x_j\right\|_{\mathscr{X}} = \|F\|_{\mathscr{X}^*}\left(\sum_j(\beta_j-\alpha_j)\right)^{\frac{1}{p}}\end{aligned} \tag{4}$$

Obviously, this implies $g(t) \in AC(I;X^*)$.

Step 3: By previous step and the fact that $g(0) = 0$, we see that (1) holds with some $v(t) \in L^1(I;X^*)$. It remains to establish (2).
If $p = 1$, observe that (4) implies $g(t)$ is lipschitz, and in particular $v(t) = g'(t)$ a. e. is essentially bounded by $\|F\|_{\mathscr{X}^*}$. In other words, (2) holds with $p' = \infty$ as required.
If $p \in (1,\infty)$, one can proceed as follows. As in Step 1 we use linearity to deduce (3) for all $u(t) = \sum_j \chi_{(\alpha_j,\beta_j]}(t)x_j$. We now just have $v(t) \in L^1(I;X^*)$, so the density argument only extends to $u(t) \in L^\infty(I;X)$.
We need one more limiting argument: set

$$v_n(t) = \begin{cases} v(t), & \text{if } \|v(t)\|_{X^*} \le n \\ 0, & \text{otherwise}\end{cases}$$

and $u_n(t) = z(t)\|v_n(t)\|_{X^*}^{p'-1}$, where $z(t) \in X$ are such that $\|z(t)\|_X = 1$ and $\langle v(t), z(t)\rangle_{X^*,X} = \|v(t)\|_{X^*}$. Now $u_n(t)$ are essentially bounded, and $\|u_n(t)\|_X^p = \langle v(t), u_n(t)\rangle_{X^*,X} = \|v_n(t)\|_{X^*}^{p'}$. Plugging $u_n(t)$ into (3) gives, after a simple manipulation, that

$$\left(\int_I \|v_n(t)\|_{X^*}^{p'}\,dt\right)^{\frac{1}{p'}} \le \|F\|_{\mathscr{X}^*}$$

Since $\|v_n(t)\|_{X^*} \nearrow \|v(t)\|_{X^*}$, estimate (2) follows by Levi theorem.

Corollary: X reflective, separable, $p \in (1,+\infty) \implies L^p(I;X)$ is reflexive and separable. Hence any sequence bounded in $L^p(I;X)$ has a weakly convergent subsequence.

1.4. More on weakly differentiable functions: extensions, approximations, embeddings

Recall: $u(t), g(t) \in L^1(I;X) \ldots$ $g(t)$ is a weak derivative of $u(t)$ $\left(\frac{d}{dt}u(t) = g(t)\right)$ iff (by L. 1.3) $\exists x_0 \in X$ such that $u(t) = x_0 + \int_0^t g(s)\,ds$ for a. e. $t \in I = [0,T]$ or $\iff$ $\int_I u(t)\varphi'(t)\,dt = -\int_I g(t)\varphi(t)\,dt$ for $\forall\varphi(t) \in D(I) = C_c^\infty(I,\mathbb{R})$.

Remark:

- by the above results: $u(t)$ is weakly differentiable $\iff$ $\exists\widetilde{u}(t) \in AC$ such that $\widetilde{u}(t) = u(t)$, $\widetilde{u}'(t) = \frac{d}{dt}u(t)$ a. e.
- in applications we will often have $u(t) \in L^p(I;Y)$, $\frac{d}{dt}u(t) \in L^q(I;Z)$. This means: $\exists X$ such that $Y, Z \subset X$ (often $Y \subset Z = X$). Then $\frac{d}{dt}u(t) = g(t)$ in $L^1(I;X)$ and moreover $u(t) \in L^p(I;Y)$, $g(t) \in L^q(I;Z)$.

Lemma 1.4 [Weak derivative of product and convolution]. Let $u(t): I \to X$ be weakly differentiable. Then:

(1) If $\eta(t) : I \to \mathbb{R}$ is C^∞,[1] then $u(t)\eta(t) : I \to X$ is weakly differentiable and $\frac{d}{dt}(u(t)\eta(t)) = \left(\frac{d}{dt}u(t)\right)\eta(t) + u(t)\eta'(t)$.
(2) If $\psi(t) \in D(I)$, then $u * \psi(t) : I \to X$ is smooth and moreover $(u*\psi)'(t) = \left(\frac{d}{dt}u\right) * \psi(t)$, whenever $t - \operatorname{supp}\psi \subset (0,T)$.

Proof:

(1) $\varphi(t) \in D(I)$ given:

$$\begin{aligned}\int_I [u(t)\eta(t)]\varphi'(t)\,dt &= \int_I u(t)\underbrace{\eta(t)\varphi'(t)}_{\substack{\| \text{ (smooth)} \\ (\eta(t)\varphi(t))' - \eta'(t)\varphi(t)}}dt \\ &= \underbrace{\int_I u(t)\underbrace{(\eta(t)\varphi(t))'}_{\in D(I)}dt}_{=-\int_I g(t)\eta(t)\varphi(t)\,dt,\ g(t)=\frac{d}{dt}u(t)} - \int_I u(t)\eta'(t)\varphi(t)\,dt \\ &= -\int_I \left[\frac{d}{dt}u(t)\eta(t) + u(t)\eta'(t)\right]\varphi(t)\,dt\end{aligned}$$

(2) extend $u(s) = 0$ for $s \notin I$ (not neccessary if $t - \operatorname{supp}\psi \subset I$). Let $t \in I$ be fixed.

$$\begin{aligned} u*\psi(t) &= \int_{\mathbb{R}} u(t-s)\psi(s)\,ds \overset{\text{subst.}}{=} \int_{\mathbb{R}} u(s)\psi(t-s)\,ds \\ &= \int_I u(s)\psi(t-s)\,ds \\ \text{we know: } (u*\psi)'(t) &= \int_I u(s)\underbrace{\psi'(t-s)}_{=-\varphi'(s),\text{ where } \varphi(s)=\psi(t-s)}ds \\ &= \int_I g(s)\varphi(s)\,ds = \int_I \frac{d}{dt}u(s)\psi(t-s) = PS\end{aligned}$$

Theorem 1.10 [Extension operator for weak derivative]. Let $u(t) \in L^p(I,Y)$, $\frac{d}{dt}u(t) \in L^q(I;Z)$, where $I = [0,T]$. Denote $I_\Delta = [-\Delta, T+\Delta]$ with $\Delta > 0$. Then there is $Eu(t) \in L^p(I_\Delta;Y)$, $\frac{d}{dt}Eu \in L^q(I_\Delta, Z)$ and such that $Eu(t) = u(t)$, $\frac{d}{dt}Eu(t) = \frac{d}{dt}u$ for a. a. $t \in I$. Moreover, $\|Eu(t)\|_{L^p(I_\Delta,Y)} \le C\|u(t)\|_{L^p(I_\Delta,Y)}$ and $\left\|\frac{d}{dt}Eu(t)\right\|_{L^q(I_\Delta,Y)} \le C\left\|\frac{d}{dt}u(t)\right\|_{L^q(I_\Delta,Y)}$ for suitable $C > 0$.

[1] Holds for lipschitz, proven for C^∞.

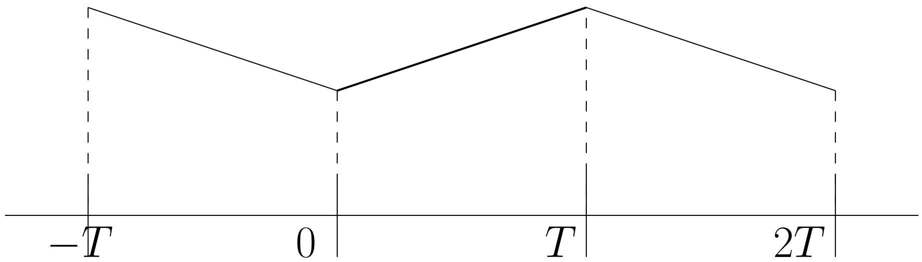

FIGURE 1.4.1. Even extension

Proof: **Trick:** use even extension as on Fig. 1.4.1: $\widetilde{u}(t)=\begin{cases}u(t) & t\in[0,T]\\ u(-t) & t\in[-T,0)\end{cases}$.
Claim: $\widetilde{u}(t)$ is weakly differentiable and $\frac{d}{dt}\widetilde{u}(t)=\widetilde{g}(t)$ in $[-T,T]$,
where $\widetilde{g}(t)=\begin{cases}g(t) & t\in[0,T]\\ -g(-t) & t\in[-T,0)\end{cases}$ with $g(t)=\frac{d}{dt}u(t)$.
Note: $t\in[-T,0]\implies$

$$\begin{aligned}\widetilde{u}(t) &\overset{\text{L. 1.3, 1}}{=} x_0+\int_0^{-t}g(s)\,ds=\underbrace{x_0+\int_0^T g(s)\,ds}_{\widetilde{x}_0\in X}-\underbrace{\int_{-t}^T g(s)\,ds}_{\text{subst. } s=-\tau}\\ &= \widetilde{x}_0-\int_{-T}^t\widetilde{g}(-\tau)\,d\tau=\widetilde{x}_0+\int_{-T}^t\widetilde{g}(s)\,ds,\end{aligned}$$

hence we have shown: $\widetilde{u}(t)=\widetilde{x}_0+\int_{-T}^t\widetilde{g}(s)\,ds$, $t\in[-T,0)$, but also $u(t)=x_0+\int_0^t g(s)\,ds$, therefore $\widetilde{u}(t)=\widetilde{x}_0+\int_{-T}^t\widetilde{g}(s)\,ds$, $t\in[-T,T]$. Clearly $\|\widetilde{u}(t)\|_{L^p([-T,2T];Y)}=3\|u\|_{L^p(I;Y)}$, $\left\|\frac{d\widetilde{u}}{dt}\right\|_{L^q([-T,2T],Z)}=3\left\|\frac{d}{dt}u\right\|_{L^q(I;Z)}$.

Remarks:

- extension operator $u\mapsto Eu$ is linear,
- we can have $Eu(t)=0$ for t outside $I_{\Delta/2}$ (multiply by cutoff function $\eta(t)$: L. 1.4,1).

Theorem 1.11 [Smooth approximation of weakly differentiable functions]. Let $u(t)\in L^p(I;Y)$, $\frac{d}{dt}u(t)\in L^q(I;Z)$. Then there exist functions $u_n(t)\in C^1(I;Y)$ such that $u_n(t)\to u(t)$ in $L^p(I;Y)$, $u_n'(t)\to\frac{d}{dt}u(t)$ in $L^q(I;Z)$.

Proof: Set $u_n(t)=Eu*\psi_n(t)$, where $\psi_n(t)=n\psi_0(nt)$, $\psi_0(t)\in C_c^\infty(\mathbb{R},\mathbb{R})$, $\int_{\mathbb{R}}\psi_0(t)\,dt=1$, $\psi_0(t)=0$ outside $[-1,1]$. $Eu(t)$ is defined for $t\in[-\Delta,T+\Delta]$ … from Th. 1.10, set $Eu(t)=0$ elsewhere.
Observe: $u_n(t)\in C^1(I,Y)$ … ex. 1.3.1, $u_n(t)\to Eu(t)$, $n\to\infty$ in $L^p(I_\Delta,Y)$ … L 1.1,4 and $Eu(t)=u(t)$ in $[0,T]$, hence $u_n(t)\to u(t)$ in $L^p(I;Y)$. Moreover, $u_n'(t)=\left(\frac{d}{dt}Eu\right)*\psi_n(t)\to\left(\frac{d}{dt}Eu(t)\right)*\psi_n$ in $L^q(I;Z)$ by L. 1.1,4 and $\frac{d}{dt}Eu(t)=\frac{d}{dt}u(t)$ in I, hence $u_n'(t)\to\frac{d}{dt}u(t)$ in $L^q(I;Z)$.

Def.: [Gelfand triple]. Let X be separable, reflexive, densely embedded into a Hilbert space H. Then, by Gelfand triple we mean $X\subset H=H^*\subset X^*$.

Commentary: Riesz: $H=H^*$ (identification): $\forall y\in H^*\ \exists!\widetilde{y}\in H$ such that $\langle y,x\rangle_{H^*,H}=(\widetilde{y},x)_H\ \forall x\in H$, where $(\cdot,\cdot)_H$ is scalar product in H. $X\subset H\implies H^*\subset X^*$, in particular $X\subset X^*$ via the embedding: $x_0\mapsto x_0^*$, where $\langle x_0^*,x\rangle_{X^*,X}=(x_0,x)_H\ \forall x\in X$. Can be shown: injective and on the dense set, see ex. 3.3.

Application: we will often work with $u(t)\in L^p(I;X)$, $\frac{d}{dt}u(t)\in L^q(I;X^*)$.

Lemma 1.5 [Weak representative for $W^{1,p}(I;X)$]. $W^{1,p}(I;X)\subset C(I,X)$ in the sense of representative: for any $u(t)\in W^{1,p}(I,X)$ there is $\widetilde{u}(t)\in C(I,X)$ such that $\widetilde{u}(t)=u(t)$ a. e. and $\|\widetilde{u}(t)\|_{C(I,X)}\le c\|u(t)\|_{W^{1,p}(I;X)}$.

Remark: One even has $W^{1,p}(I;X)\hookrightarrow C^{0,\alpha}(I,X)$, where $\alpha=1-\frac{1}{p}$ (HW 2).

Proof: **Step 1:** $u(t) \in C^1(I,X)$ ... fix $t \in I$, WLOG let $t \in \left[\frac{T}{2}, T\right]$, where $I = [0,T]$. Then $u(t) = u(\tau) + \int_\tau^t \frac{d}{ds} u(s)\, ds$ for $\tau \in \left[0, \frac{t}{2}\right]$. Therefore

$$\begin{aligned}
\|u(t)\| &\leq \|u(\tau)\| + \int_I \left\| \frac{d}{dt} u(s) \right\| ds \quad / \int_0^{T/2} d\tau \\
\frac{T}{2} \|u(t)\| &\leq \int_0^{\frac{T}{2}} \|u(\tau)\|\, d\tau + \frac{T}{2} \int_I \left\| \frac{d}{dt} u(s) \right\| ds \\
&\leq c \|u(t)\|_{W^{1,1}(I;X)} \leq \widetilde{c} \|u\|_{W^{1,p}(I;X)} \\
&\implies \|u(t)\|_{C(I;X)} \leq c \|u(t)\|_{W^{1,p}(I;X)} \quad (\#)
\end{aligned}$$

$c = c(T)$ for any $u(t) \in C^1(I,X)$

Step 2: $u(t) \in W^{1,p}(I;X)$ arbitrary. Th. 1.11 $\implies \exists u_n(t) \in C^1(I;X)$, $u_n(t) \to u(t)$ in $W^{1,p}(I;X)$. Hence $\{u_n(t)\}$ is cauchy in $W^{1,p}(I;X)$ and we apply $(\#)$ to $u_n(t) - u_m(t) \implies \{u_n(t)\}$ is cauchy in $C(I,X)$, hence $u_n(t) \rightrightarrows \widetilde{u}(t)$, where $\widetilde{u}(t) \in C(I,X)$ is the continuous representative.

Theorem 1.12 [Continuous representative for $L^p(I;X)$ with $L^{p'}(I,X^*)$ derivative]. Let $X \subset H = H^* \subset X^*$ be Gelfand triple. Let $u(t) \in L^p(I;X)$, $\frac{d}{dt} u(t) \in L^{p'}(I;X^*)$, where p, p' are Hölder conjugate. Then

(1) $u(t) \in C(I,H)$ in the sense of representative:
$\exists \widetilde{u}(t) \in C(I,H)$ such that $\|\widetilde{u}\|_{C(I,H)} \leq c\left(\|u\|_{L^p(I;X)} + \left\|\frac{d}{dt} u\right\|_{L^{p'}(I;X^*)}\right)$ and $\widetilde{u}(t) = u(t)$ a. e. in I.

(2) $t \mapsto \|u(t)\|_H^2$ is weakly differentiable with $\frac{d}{dt}\|u(t)\|_H^2 = 2\left\langle \frac{d}{dt} u(t), u(t)\right\rangle_{X^*,X}$ a. e. in I, in particular, $\|\widetilde{u}(t_2)\|_H^2 = \|\widetilde{u}(t_1)\|_X^2 + \int_{t_1}^{t_2} 2\left\langle \frac{d}{dt} u(s), u(s)\right\rangle_{X^*,X} ds$ where $\widetilde{u}(t)$ is the continuous representative.

Proof:

(1) **Step 1:** $u(t) \in C^1(I,X)$. **Trick:** $u(t) = \underbrace{u(t)\theta(t)}_{u_1(t)} + \underbrace{u(t)(1-\theta(t))}_{u_2(t)}$, $\theta(t): I \to \mathbb{R}$ is smooth, $\theta(0) = 0$, $\theta(T) = 1$, e. g. $\theta(t) = \frac{t}{T}$.

$$\begin{aligned}
\|u_1(t)\|_H^2 &= (u_1(t), u_1(t))_H - \underbrace{(u_1(0), u_1(0))_H}_{0} \\
&= \int_0^t \frac{d}{ds}(u_1(s), u_1(s))\, ds \\
&= \int_0^t 2\Big(\underbrace{u_1'(s)}_{u'(s)\theta(s) + u(s)\theta'(s)}, \underbrace{u_1(s)}_{u(s)\theta(s)}\Big) ds \\
&= 2\int_0^t (u'(s), u(s))_H\, \theta^2(s) + (u(s), u(s))_H\, \theta'(s)\theta(s)\, ds \\
\begin{pmatrix}\text{Gelfand emb.} \\ \iota: X \to X^*\end{pmatrix} &= 2\int_0^t \langle \iota u'(s), u(s)\rangle_{X^*,X}\, \theta(s) + \langle \iota u(s), u(s)\rangle_{X^*,X}\, \theta'(s)\theta(s)\, ds \\
&\leq C\int_I |\langle \iota u'(s), u(s)\rangle| + |\langle \iota u(s), u(s)\rangle|\, ds
\end{aligned}$$

"generalized scalar product": Gelfand: $\langle \iota u, v\rangle_{X^*,X} = (u,v)_H$ ... use Hölder

$$\begin{aligned}
&\leq C\left(\|\iota u'\|_{L^{p'}(I;X)} \|u\|_{L^p(I;X)}\right) + \underbrace{\|\iota u\|_{L^\infty(I;X^*)}}_{\overset{\text{L. 1.1}}{\leq} C\|u\|_{W^{1,1}(I;X^*)}} \|u\|_{L^1(I;X)} \\
&\leq C\left(\|u\|_{L^p(I;X)} + \left\|\frac{d}{dt}\iota u\right\|_{L^{p'}(I;X)}\right)^2
\end{aligned}$$

And therefore part 1 holds for $u(t) \in C^1(I,X)$, namely we have:
(☼) $\sup_{t\in[0,T]} \|u(t)\|_H \le c\left(\|u\|_{L^p(I;X)} + \left\|\frac{d}{dt}u\right\|_{L^p(I;X^*)}\right)$.
Step 2: let $u(t) \in L^p(I;X)$ with $\frac{d}{dt}u(t) \in L^{p'}(I;X^*)$ be given. By Th. 1.11 there exists $u_n \in C^1(I;X)$ such that $u_n(t) \to u(t)$ in $L^p(I;X)$ and $\iota u_n'(t) \to \frac{d}{dt}u(t)$ in $L^{p'}(I;X^*)$. Apply (☼) to $u_n(t) - u_m(t) \in C^1(I,X)$, hence RHS is small for $m,n \ge n_0$ for n_0 large enough. Therefore LHS is small $\implies (B.C.)$ for $u_n(t): I \to H \implies \exists \widetilde{u}(t) \in C(I,H)$ such that $u_n(t) \rightrightarrows \widetilde{u}(t)$ in I. Also: $u_n(t) \to u(t)$ a. e. in X, hence $u(t) = \widetilde{u}(t)$ a. e., where $\widetilde{u}(t)$ is the representative.

(2) Let $u_n(t)$ be smooth approximations:

$$\begin{aligned}\frac{d}{dt}\|u_n(t)\|_H^2 &= \frac{d}{dt}(u_n(t), u_n(t)) = 2(u_n'(t), u_n(t)) \\ &= 2\langle \iota u_n'(t), u_n(t)\rangle_{X^*,X}\end{aligned}$$

For $\forall t_1, t_2 \in I$ we have

$$\begin{aligned}\|u_n(t_2)\|_H^2 &= \|u_n(t_1)\|_H^2 + 2\int_{t_2}^{t_1} \langle \iota u_n'(s), u_n(s)\rangle_{X^*,X}\, ds \\ &\downarrow \text{n}\to\infty \\ \|\widetilde{u}(t_2)\|_H^2 &= \|\widetilde{u}(t_1)\|_H^2 + 2\int_{t_2}^{t_1} \langle \iota u'(s), u(s)\rangle_{X^*,X}\, ds\end{aligned}$$

by Hölder inequality (ex.: $u_n(t) \to u(t)$ in $L^p(I;X)$, $v_n(t) \to v(t)$ in $L^{p'}(I;X^*)$ which implies the convergence $\int_I \langle v_n(s), u_n(s)\rangle\, ds \to \int_I \langle v(s), u(s)\rangle\, ds$, see ex. 2.5.

Notation:

embedding: $X \hookrightarrow Y$: $X \subset Y$, $\exists c > 0$ such that $\|u\|_Y \le c\|u\|_X$
compact embedding: $X \hookrightarrow\hookrightarrow Y$: $X \hookrightarrow Y$ and for any $\{u_n\} \subset X$ bounded there is a strongly convergent (in Y) subsequence.

Remarks:

- Th. 1.12: $\left\{u(t) \in L^p(I;X), \frac{d}{dt}u(t) \in L^{p'}(I;X)\right\} \hookrightarrow C(I,H)$ in the sense of representative.
- Gelfand triple: $X \hookrightarrow H \cong H^* \hookrightarrow X^*$, also $X \hookrightarrow X^*$, more precisely: the embedding of X into X^* is given by $\iota X \to X^*$, where $\langle \iota u, v\rangle_{X^*,X} = (u,v)_H$, $u,v \in X$, because: $\frac{d}{dt}u(t) = g(t) \iff \int_I \iota u(t)\varphi'(t)\,dt = -\int g(t)\varphi(t)\,dt\ \forall\varphi(t) \in D(I)$.

Lemma 1.6 [Ehrling lemma]. Let $Y \hookrightarrow\hookrightarrow X \hookrightarrow Z$. Then for any $a > 0$ there is $C > 0$ such that $\|u\|_X \le \alpha\|u\|_Y + C\|u\|_Z$.

Proof: for contradiction assume that $\forall n\ \exists u_n$ such that $\|u_n\|_X > a\|u_n\|_Y + n\|u_n\|_Z$, WLOG $\|u_n\|_X = 1$, hence $1 > a\|u_n\|_Y + n\|u_n\|_Z$. Therefore $\{u_n\}$ is bounded in $Y \implies \exists$subsequence $\widetilde{u}_n \to u \in X$, $\|u\|_X = 1$. But: $\frac{1}{n} > \|u_n\|_Z$, $u_n \to 0$ in Z. Therefore $u = 0$ in Z and thus we have obtained a contradiction: $u = 0$ in $Z \implies u = 0$ in X since $X \hookrightarrow Z$

Theorem 1.13 [Aubin–Lions lemma]. Let $Y \hookrightarrow\hookrightarrow X \hookrightarrow Z$, where X, Z are reflexive and separable. Let $p,q \in (1,\infty)$. Then for any sequence $u_n(t)$ bounded in $L^p(I;Y)$ such that $\frac{d}{dt}u_n(t)$ are bounded in $L^q(I;Z)$ there is a subsequence converging strongly in $L^p(I;X)$.
(Briefly: $\left\{u(t) \in L^p(I,Y),\ \frac{d}{dt}u(t) \in L^q(I;Z)\right\} \hookrightarrow\hookrightarrow L^p(I;X)$.)

Proof: **Step 1:** $L^p(I;Y)$, $L^q(I;Z)$ are reflexive, by Eberlein–Šmulian $\exists$ subsequence such that $u_n(t) \rightharpoonup u(t)$ in $L^p(I;Y)$ and $\frac{d}{dt}u_n(t) \rightharpoonup g(t)$ in $L^q(I;Z)$ and $\frac{d}{dt}u(t) = g(t)$. We will show: $u_n(t) \to u(t)$ in $L^p(I;X)$. WLOG $u(t) = 0\ \ldots$ otherwise subtract the limit (see ex. 2.1).
Subproblem: $u_n(t) \rightharpoonup 0$ in $L^p(I;Y)$ and $\frac{d}{dt}u_n(t) \rightharpoonup 0$ in $L^q(I;Z) \implies u_n(t) \to 0$ in $L^p(I;X)$.

Step 2: $u_n \to 0$ (strongly) in Z for any $t \in I$ fixed. **Trick:** $u_n(t) = u_n(t+s) - \int_t^{t+s} \frac{d}{dt} u_n(\sigma)\, d\sigma$, $s \in (t, t+\delta)$. Integrate $\int_0^\delta \cdot ds$, multiply by $\frac{1}{\delta}$:

$$u_n(t) = \underbrace{\frac{1}{\delta}\int_0^\delta u_n(t+s)\, ds}_{I_{1n}} - \underbrace{\frac{1}{\delta}\int_0^\delta \int_t^{t+s} u_n(\sigma)\, d\sigma ds}_{I_{2n}}$$

I_{1n}: recall: $x_n \rightharpoonup 0$ in X, $L : X \to Y$ linear continuous $\implies Lx_n \rightharpoonup 0$ in Y (see "Properties of weak convergence" in appendices). Observe: $u \mapsto \frac{1}{\delta}\int_0^\delta u(t+s)\, ds$ is linear continuous $L^p(I;Y) \to Y \hookrightarrow\hookrightarrow Z$. Recall: $x_n \rightharpoonup 0$ in Y, $Y \hookrightarrow\hookrightarrow Z \implies x_n \to 0$ in Z. Hence $I_{1n} \to 0$ in Z.
I_{2n}:

$$\begin{aligned}
\int_0^\delta \left(\int_t^{t+s} \frac{d}{dt} u_n(\sigma)\, d\sigma\right) ds &\overset{\text{Fub.}}{=} \int_t^{t+\delta} \left(\int_{\sigma - t}^{\sigma} \frac{d}{dt} u_n(\sigma)\, ds\right) d\sigma \\
&= \int_t^{t+\delta} (\sigma - \sigma + t) \frac{d}{dt} u_n(\sigma)\, d\sigma
\end{aligned}$$

$$\begin{aligned}
\implies \|I_{2n}\|_Z &\leq \int_t^{t+\delta} \left\|\frac{d}{dt} u_n(\sigma)\right\|_Z d\sigma \\
\text{(Hölder)} &\leq \left(\int_t^{t+\delta} \left\|\frac{d}{dt} u_n(\sigma)\right\|_Z^q d\sigma\right)^{\frac{1}{q}} \left(\int_t^{t+\delta} 1^{q'} d\sigma\right)^{\frac{1}{q'}} \\
&\leq \left\|\frac{d}{dt} u_n\right\|_{L^q(I;Z)} \delta^{1-\frac{1}{\delta}} \leq K\delta^{1-\frac{1}{\delta}}
\end{aligned}$$

where $1 - \frac{1}{q} > 0$ since $q > 1$.
Therefore for $\varepsilon > 0$ given choose $\delta > 0$ small such that $\|I_{2n}\|_Z \leq \frac{\varepsilon}{2}$ for $\forall n$. Then choose n_0 such that $\|I_{1n}\|_Z < \frac{\varepsilon}{2}$ if $n > n_0 \implies \|u_n(t)\|_Z < \varepsilon$ for $n > n_0$.
Step 3: $u_n(t) \to 0$ in $L^p(I;X)$, $\varepsilon > 0$ given. Use L. 1.6: $\|u_n(t)\|_X \leq a\|u_n(t)\|_Y + C\|u_n(t)\|_Z$. Δ-inequality: $\|u_n\|_{L^p(I;X)} \leq a\|u_n\|_{L^p(I;Y)} + c\|u_n\|_{L^p(I;Z)}$. Choose $a = \frac{\varepsilon}{2k}$, where $\|u_n\|_{L^p(I;Y)} \leq K$. But: $\|u_n(t)\|_{L^p(I;Z)} = \left(\int_I \|u_n(t)\|_Z^p\, dt\right)^{\frac{1}{p}} \to 0$ by Lebesgue. Since $\|u_n(t)\|_Z \to 0$ by step 2 and $\|u_n(t)\|_Z \leq C$ by embedding $W^{1,1}(I;Z) \hookrightarrow C(I,Z)$ is u_n is bounded in $W^{1,1}(I;Z)$, hence second term $< \frac{\varepsilon}{2}$ for n large, therefore $\|u_n\|_{L^p(I;X)} \leq \varepsilon$ for n large.

CHAPTER 2

Parabolic second order equations

$$
\begin{aligned}
(P1)\quad \partial_t u - \operatorname{div} a(\nabla u) + f(u) &= h(t,x), \quad (t,x) \in I \times \Omega \\
(P2)\quad u &= u_0, \quad t = 0,\ x \in \Omega \\
(P3)\quad u &= 0, \quad x \in \partial\Omega
\end{aligned}
$$

where $\nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_2}\right)$; operators Δ and div are defined as usual. $u = u(t,x)$ is unknown solution, $h = h(t,x)$ is right–hand side, $u_0 = u_0(x)$ is initial condition. h and u_0 define the data of the equation.

Assumptions.

($A1$) $\Omega \in \mathbb{R}^n$ is bounded domain (i. e., open, connected) with regular (Lipschitz) boundary.
($A2$) $a(\xi) : \mathbb{R}^n \to \mathbb{R}^n$, $a(0) = 0$, $|a(\xi_1) - a(\xi_2)| \leq \alpha_1 |\xi_1 - \xi_2|$, $(a(\xi_1) - a(\xi_2)) \cdot (\xi_1 - \xi_2) \geq \alpha_0 |\xi_1 - \xi_2|^2$ $\forall \xi_1, \xi_1 \in \mathbb{R}^n$.
($A3$) $f(z) : \mathbb{R} \to \mathbb{R}$, $|f(z_1) - f(z_2)| \leq \ell |z_1 - z_2|$, $\forall z_1, z_2 \in \mathbb{R}$.

Remark:

- special case: $a(\xi) = \xi$, $f = 0$ ... heat equation $\partial_t u - \Delta u = h(t,x)$.

Plan:

(1) well–posedness ($\exists!$ weak solution)
(2) regularity (strong solution for smooth data)
(3) further properties

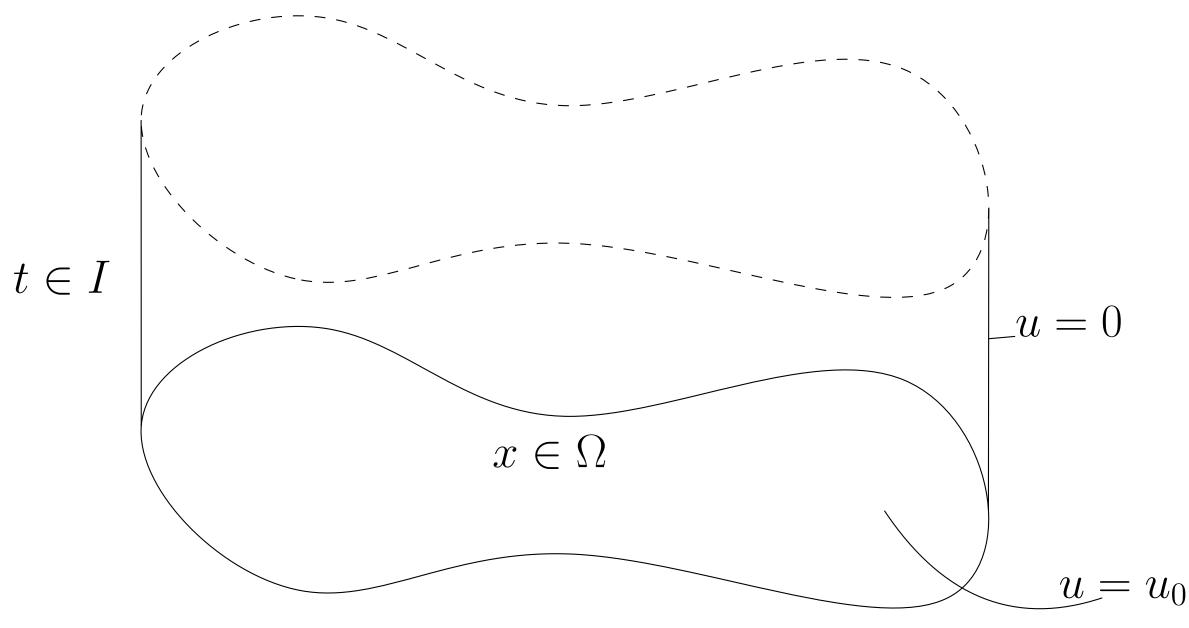

Figure 2.0.1. Initial and boundary condition in spacetime cylinder.

Recall:

- Sobolev spaces $W^{1,2} = \left\{u \in L^2(\Omega) : \nabla u = v \in L^2(\Omega)\right\}$, where ∇u is weak derivative, i. e., $-\int_\Omega u \frac{\partial \varphi}{\partial x_j}\, dx = \int_\Omega v_j \varphi\, dx\ \forall \varphi \in D(\Omega), \forall j$.
  $\|u\|_{W^{1,2}(\Omega)} = \|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}$
- $W_0^{1,2}(\Omega) = \left\{u \in W^{1,2}(\Omega), u = 0 \text{ on } \partial\Omega \text{ in the sense of traces}\right\} = \overline{C_c^\infty(\Omega)}^{W^{1,2}(\Omega)}$
- $W^{-1,2}(\Omega) = \left(W_0^{1,2}(\Omega)\right)^*$

Facts:

- compact embedding: $W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$
- Poincaré inequality: $\|u\|_{L^2(\Omega)} \le c_p \|\nabla u\|_{L^2(\Omega)}$ for $\forall u \in W_0^{1,2}(\Omega)$.
  Corollary: norms $\|u\|_{W^{1,2}(\Omega)}$ and $\|\nabla u\|_{L^2(\Omega)}$ are equivalent on $W_0^{1,2}(\Omega)$.
- $W^{1,2}(\Omega)$ is reflexive, separable, $W_0^{1,2}(\Omega)$ is a closed subspace of $W^{1,2}(\Omega)$ hence also reflexive, separable.
- $W_0^{1,2}(\Omega)$, $W^{1,2}(\Omega)$ are dense in $L^2(\Omega)$.

Notation: we will write L^2, $W^{1,2}$, $W_0^{1,2}$ instead of $L^2(\Omega)$, $\|u\|_2$, $\|u\|_{1,2}$, $\|u\|_{-1,2}$ instead of $\|u\|_{L^2(\Omega)}$ etc. $(\cdot,\cdot)$ is scalar product in $L^2(\Omega)$, i. e., $(f,g) = \int_\Omega \underbrace{f(x)}_{}\cdot \underbrace{g(x)}_{}\, dx$, $\langle\cdot,\cdot\rangle$ is duality of $W^{-1,2}$ and $W_0^{1,2}$.
both functions in $\mathbb{R}^n$ or $\mathbb{R}$
Gelfand triple: $W_0^{1,2} \hookrightarrow L^2 \cong \left(L^2\right)^* \hookrightarrow W^{-1,2}$ with the embedding $\iota : L^2 \hookrightarrow L^2 : \langle \iota u, v\rangle = (u,v)$, $\forall v \in L^2$.

Recall: stationary problem:

$$\begin{aligned} -\mathrm{div} a(\nabla u) &= h, \quad x \in \Omega, h \in W^{-1,2} \\ u &= 0, \quad x \in \partial\Omega \end{aligned}$$

weak solution: $u \in W_0^{1,2}$ such that: $\int_\Omega a(\nabla u(x)) \cdot \nabla v(x)\ dx = \langle h, v\rangle$ for $\forall v \in W_0^{1,2}$

- by nonlinear Lax–Milgram theorem: $\forall h \in W^{-1,2}\ \exists! u \in W_0^{1,2}$ weak solution
- we introduce operator $A : W_0^{1,2} \to W^{-1,2}$ by $\langle A(u), v\rangle = (a(\nabla u), v)$. By Lax–Milgram: $A : W_0^{1,2} \to W^{-1,2}$ is continuous, 1-to-1.

Remark: if $a(\xi) = \xi$, then $\langle A(u), v\rangle = (\nabla u, \nabla v)$, i. e., $A(u) = -\Delta u$ (weak laplacian).

Assumption: $h(t) \in L^2\left(I; W^{-1,2}\right)$, where $h(t) = h(t,x)$ is the RHS in $(P1)$.

Def.: [Weak solution of parabolic equation]. function $u(t) \in L^2\left(I; W_0^{1,2}\right)$ is called a weak solution to $(P1)$–$(P3)$ provided that:

$$\frac{d}{dt}(u(t), v) + (a(\nabla u(t)), \nabla v) + (f(u(t)), v) = \langle h(t), v\rangle$$

in the sense of distributions in $(0,T)$ for $\forall v \in W_0^{1,2}$. ("Infinite system of ODEs.")

Remark: Expanding the definition of $\frac{d}{dt}$ we get

$$\frac{d}{dt} y(t) = g(t) \iff \int_I y(t)\varphi'(t) = -\int_I g(t)\varphi(t)\, dt\ \forall \varphi(t) \in D(I)$$

$$\begin{aligned} \Longrightarrow \quad (*)\ & -\int_I (u(t), v)\varphi'(t)\, dt + \int_I (a(\nabla u(t)), \nabla v)\varphi dt \\ & + \int_I (f(u(t)), v)\varphi(t)\, dt = \int_I \langle h(t), v\rangle \varphi(t)\, dt \\ & \text{for } \forall v \in W_0^{1,2}, \forall \varphi(t) \in D(I) \end{aligned}$$

by expanding further $(\cdot,\cdot)$ we get

$$-\iint\limits_{I\times\Omega} u(t,x)\,v(x)\,\varphi'(t)\,dt\,dx+\iint\limits_{I\times\Omega} a(\nabla u(t,x))\cdot\nabla v(x)\,\varphi(t)\,dt\,dx$$
$$+\iint\limits_{I\times\Omega} f(u(t,x))\,v(x)\,\varphi(t)\,dt\,dx=\int_I\langle h(t),v\rangle\,\varphi(t)\,dt$$
$$\text{for }\forall v\in W_0^{1,2}$$

Note:

- multiply $(P1)$ by $v(x)\,\varphi(t)$, integrate by parts ... formally we obtain the equation above.
- definiton makes sense: all integrals are finite (Hölder & $(A1)$, $(A2)$ ... see ex. 3)

Lemma 2.1 [Properties of weak solution of parabolic equation]. Let $u(t)\in L^2\left(I;W_0^{1,2}\right)$ be weak solution. Then

(1) $u(t)$ is weakly differentiable as a function $I\to W^{-1,2}$ with
$$(**)\qquad \frac{d}{dt}u(t)+Au(t)+\iota f(u(t))=h(t),$$
in particular $\frac{d}{dt}u(t)\in L^2\left(I,W^{-1,2}\right)$,

(2) $u(t)\in C\left(I,L^2\right)$ in the sense of representative,

(3) $t\mapsto\|u(t)\|_2^2$ is weakly differentiable with
$$(***)\qquad \frac{d}{dt}\|u(t)\|_2^2+\langle A(u(t)),u(t)\rangle+(f(u(t)),u(t))=\langle h(t),u(t)\rangle\ \text{ a. e. in } I$$

Proof:

(1) with embedding $\iota: W_0^{1,2}\to W^{-1,2}$ from Gelfand triple: $(u(t),v)=\langle\iota u(t),v\rangle$, $(f(u(t)),v)=\langle\iota f(u(t)),v\rangle$, by definition: $(a(\nabla u(t)),\nabla v)=\langle A(u(t)),v\rangle$. But then $(*)$ is rewritten as
$$\left\langle\underbrace{\int_I -\iota u(t)\,\varphi'(t)+A(u(t))\,\varphi(t)+\iota f(u(t))\,\varphi(t)-h(t)\,\varphi(t)\,dt}_{\substack{\|\\ 0}\ \text{since eq. holds }\forall\varphi(t)\in D(I),\ \forall v\in W_0^{1,2}},v\right\rangle=0.$$
Therefore $(**)$ is proven.
$\frac{d}{dt}u(t)\in L^2\left(I,W^{-1,2}\right)$ follows from $(**)$ and the fact that $A(u(t))$, $f(u(t))$, $h(t)$ belong to this space.

(2) Apply Th. 1.12,1 with $X=W_0^{1,2}$, $X^*=W^{-1,2}$, $H=L^2$, $p=p'=2$.

(3) Apply Th. 1.12,2 with $X=W_0^{1,2}$, $X=W_0^{1,2}$, $X^*=W^{-1,2}$, $H=L^2$, $p=p'=2$ ... $\frac{d}{dt}\|u(t)\|_2^2=2\left\langle\frac{d}{dt}u(t),u(t)\right\rangle\overset{\text{use }(**)}{=}\dots$

Remark:

- we will always work with continuous representative, hence $u(t)$ is well-defined for all $t\in I$, in particular initial condition $u(0)=u_0$ makes sense,
- also, $t\to\|u(t)\|_2^2$ is AC (Th. 1.12).

Theorem 2.1 [Uniqueness of weak solution of parabolic equation]. Weak solution is unique.

Proof: Let $u(t),v(t)\in L^2\left(I;W_0^{1,2}\right)$ be weak solutions and assume $u(0)=v(0)$. Set $w(t)=u(t)-v(t)$. Goal: $w(t)=0\ \forall t\in I$.
By L. 2.1: $w(t)$ is weakly differentiable with $\frac{d}{dt}w(t)+A(u(t))-A(v(t))+f(u(t))-f(v(t))=0$ in $W^{-1,2}$ for a. a. $t\in I$. Apply $\langle\cdot,w(t)\rangle$, by Th. 1.12:

$$\underbrace{\left\langle \frac{d}{dt} w(t), w(t) \right\rangle}_{\substack{\text{Th. 1.12 } \| \\ \frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2}} + \underbrace{\langle A(u(t)) - A(v(t)), w\rangle}_{(t_2)} + \underbrace{(f(u(t)) - f(v(t)), w(t))}_{(t_3)} = 0$$

$$(t_2): \quad \int_\Omega \underbrace{(a(\nabla u(t)) - a(\nabla v(t)))(\nabla u(t) - \nabla v(t))}_{\geq 0 \text{ by } (A1)} \, dx \geq 0$$

$$(t_3): \quad \left| \int_\Omega \underbrace{(f(u(t)) - f(v(t)))}_{\leq \ell |u(t) - v(t)| \text{ by } (A2)} (u(t) - v(t)) \, dx \right|$$

$$\leq \int_\Omega \ell |w(t)|^2 \, dx = \ell \|w(t)\|_2^2$$

Therefore

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2 &= \underbrace{-(t_2)}_{\leq 0} + \underbrace{(-(t_3))}_{\leq |(t_3)|} \\ &\leq 0 + \ell \|w(t)\|_2^2 \end{aligned}$$

$\implies \frac{d}{dt}\|w(t)\|_2^2 \leq 2\ell \|w(t)\|_2^2$ for a. a. $t \in I \overset{\text{L. 2.2}}{\implies} \|w(t)\|_2^2 \leq \|w(0)\|_2^2 e^{2\ell t}$ for $\forall t$, therefore $w(t) = 0$ in I.

Lemma 2.2 [Gronwall lemma]. Let $y(g)$, $g(t)$ be nonnegative (scalar) functions, $y(t)$ continuous, $g(t)$ integrable. Let $y(t) \leq K + \int_0^t g(s) y(s) \, ds$ for $\forall t \in I$. Then $y(t) \leq K \exp\left(\int_0^t g(s) \, ds\right)$ for $\forall t \in I$.

Proof: see ex. 3.

Goal: for $\forall u_0 \in L^2$, $h(t) \in L^2\left(I; W^{-1,2}\right)$ $\exists u \in L^2\left(I; W_0^{1,2}\right)$ weak solution such that $u(0) = u_0$.

Def: [Monotone and hemi-continuous operator]. $\mathscr{X}$ is Banach space, let $A : \mathscr{X} \to \mathscr{X}^*$ be (nonlinear) operator, A is

monotone: $\langle A(u) - A(v), u - v\rangle \geq 0 \ \forall u, v \in \mathscr{X}$.
hemicontinuous: $t \mapsto A(u + tv)$ is continuous ($\mathbb{R} \to \mathscr{X}$) for $\forall u, v \in \mathscr{X}$ fixed.

Lemma 2.3 [Minty's trick]. Let $A : \mathscr{X} \to \mathscr{X}^*$ be monotone, hemicontinuous. Let $u_n \rightharpoonup u$ in $\mathscr{X}$, $A(u_n) \rightharpoonup \alpha$ in $\mathscr{X}^*$. Let $\limsup_{n\to\infty} \langle A(u_n), u_n\rangle \leq \langle \alpha, u\rangle$. Then $\alpha = A(u)$, i. e., $A(u_n) \rightharpoonup A(u)$.

Proof:

$$\langle A(u_n) - A(v), u_n - v\rangle \geq 0 \quad v \in \mathscr{X} \text{ to be specified later.}$$

$$\begin{aligned} \langle A(u_n), u_n\rangle &\geq \langle A(u_n), v\rangle + \langle A(v), u_n - v\rangle \qquad \Big/ \limsup_{n\to\infty} \\ \limsup_{n\to\infty} \langle A(u_n), u_n\rangle &\geq \langle \alpha, v\rangle + \langle A(v), u - v\rangle \end{aligned}$$

using last assumptions:

$$\begin{aligned} \langle \alpha, u\rangle &\geq \langle \alpha, v\rangle + \langle A(v), u - v\rangle \\ \langle \alpha - A(v), u - v\rangle &\geq 0 \quad \text{trick: } v = u \pm \lambda w, \lambda > 0, w \in \mathscr{X} \end{aligned}$$

$$\begin{aligned} \Longrightarrow \langle \alpha - A(u \pm \lambda v), \mp\lambda w\rangle &\geq 0 \\ \Longrightarrow \langle \alpha - \underbrace{A(u \pm \lambda v)}_{\substack{\downarrow \text{ by hemicontinuity} \\ A(u)}}, w\rangle &= 0 \\ \Longrightarrow \langle \alpha - A(u), w\rangle &= 0 \quad \text{for } \forall w \in \mathscr{X} \\ \Longrightarrow \alpha - A(u) &= 0 \end{aligned}$$

Theorem 2.2 [Compactness of weak solutions of parabolic equation]. Let $u_n(t) \in L^2\left(I; W_0^{1,2}\right)$ be weak solutions such that $u_n(0) \to u_0$ in L^2. Then there is a subsequence $\widetilde{u}_n(t) \rightharpoonup u \in L^2\left(I; W_0^{1,2}\right)$ such that $u(t)$ is a weak solution with $u(0) = u_0$.

Proof: **Step 1:** a priori estimates

$$\underbrace{\frac{d}{dt}u_n(t)}_{(t_1)} + \underbrace{A(u_n(t))}_{(t_2)} + \underbrace{f(u_n(t))}_{(t_3)} = \underbrace{h(t)}_{(t_4)} \quad \text{a. e. in } I, \quad \Big/ \langle \cdot, u_n(t)\rangle$$

(t_1) $\left\langle \frac{d}{dt}u_n(t), u_n(t)\right\rangle = \frac{1}{2}\frac{d}{dt}\|u_n(t)\|_2^2$ by Th. 1.12, L. 2.1.

(t_2) below we omit (t) (argument of functions).

$$\begin{aligned} \langle A(u_n), u_n\rangle &= \int_\Omega a(\nabla u_n)\cdot \nabla u_n \, dx \\ &= \int_\Omega \Big(a(\nabla u_n) - \underbrace{a(\nabla 0)}_{=0}\Big)(\nabla u_n - \nabla 0)\, dx \\ &\overset{(A1)}{\geq} \int_\Omega \alpha |\nabla u_n - \nabla 0|^2 \, dx \\ &= \alpha \|\nabla u_n\|_2^2 \overset{\text{Poincaré}}{\geq} c_1 \|u_n\|_{1,2}^2 \end{aligned}$$

(t_3)

$$\begin{aligned} (f(u_n), u_n) &= \int_\Omega f(u_n) u_n \, dx \\ &= \int_\Omega \underbrace{(f(u_n) - f(0))(u_n - 0)}_{|\cdot| < \ell |u_n| \text{ by } (A2)} dx + f(0)\int_\Omega u_n \, dx \end{aligned}$$

$$\Longrightarrow |(t_3)| \leq \|u_n\|_1 + f(0)\cdot \|u_n\|_1 \overset{(*)}{\leq}$$

but $\|u_n\|_1 \overset{\Omega \text{ bounded}}{\leq} c\|u_n\|_2$. Young: $\|u_n\|_2 = \|u_n\|_2 \cdot 1 \leq \frac{1}{2}\|u_n\|_2^2 + \frac{1}{2}$

$$\overset{(*)}{\leq} c_3\left(1 + \|u_n\|_2^2\right) \quad \text{where } c_3 = c_3(\Omega, \ell, \ldots)$$

(t_4)

$$|(t_4)| = |\langle h(t), u_n\rangle| \leq \|h\|_{-1,2}\|u_n\|_{1,2} \overset{\text{Young}}{\leq} \frac{c_1}{2}\|u_n\|_{1,2}^2 + \frac{1}{2c_1}\|h\|_{-1,2}^2$$

Then:

$$\underbrace{\left\langle \frac{d}{dt}u_n(t), u_n(t)\right\rangle}_{=\frac{1}{2}\frac{d}{dt}\|u_n(t)\|_2^2} + \underbrace{\langle A(u_n(t)), u_n(t)\rangle}_{\geq c_1\|u_n\|_{1,2}^2} + \underbrace{\langle f(u_n(t)), u_n(t)\rangle}_{\leq c_3\left(1+\|u_n\|_2^2\right)} = \underbrace{\langle h(t), u_n(t)\rangle}_{\leq \frac{c_1}{2}\|u_n\|_{1,2}^2 + \frac{1}{2c_1}\|h\|_{-1,2}^2}$$

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|_2^2 + c_1\|u_n(t)\|_{1,2}^2 \leq$$

$$c_3\left(1+\|u_n(t)\|_2^2\right) + \frac{c_1}{2}\|u_n(t)\|_{1,2}^2 + \frac{1}{2c_1}\|h(t)\|_{-1,2}^2 \qquad \Big/ \; 2\int_0^t ds$$

$$\Longrightarrow \|u_n(t)\|_2^2 + c_1\int_0^t \|u_n(s)\|_{1,2}^2\,ds \leq \|u_n(0)\|_2^2 + 2c_3t +$$

$$\frac{1}{c_1}\int_0^t \|h(s)\|_{-1,2}^2\,ds + 2c_3\int_0^t \|u_n(s)\|_2^2\,ds \qquad \text{for } \forall t \in I.$$

Set $Y(t) = \|u_n(t)\|_2^2 + c_1\int_0^t \|u_n(s)\|_{1,2}^2\,ds$, let $K>0$ be such that $\|u_n(0)\|_2^2 + 2c_3t + \frac{1}{c_1}\int_0^t \|h(s)\|_{1,2}^2\,ds \leq K$ for $\forall t \in I$, $\forall n$. Hence we obtain $Y(t) \leq K + \int_0^t 2c_3 Y(s)\,ds$ for $\forall t \in I$. By L. 2.2 (Gronwall): $Y(t) \leq Ke^{2c_3t}$ for $\forall t \in I = [0,T]$. Therefore $Y(t)$ is bounded in I (independently of n) and so $Yu_n(t)$ is bounded in $L^\infty(I;L^2)$ and $L^2\left(I;W_0^{1,2}\right)$. Moreover $\frac{d}{dt}u_n(t)$ are bounded in $L^2\left(I;W^{-1,2}\right)$ … follows from the equation (ex. 3).

Step 2: convergent subsequences: $\exists$ subsequence (denote by $u_n(t)$) such that $u_n \rightharpoonup u(t)$ in $L^2\left(I;W_0^{1,2}\right)$ by Eberlein–Šmulian and $\frac{d}{dt}u_n(t) \rightharpoonup g(t)$ in $L^2\left(I;W^{-1,2}\right)$. By HW2 we know $g(t) = \frac{d}{dt}u(t)$. By Th. 1.13 (with $Y = W_0^{1,2}$, $X = L^2$, $Z = W^{-1,2}$, $p = q = 2$) it holds that $u_n(t) \to u(t)$ in $L^2\left(I;L^2\right)$.

Moreover: $u_n(t) \rightharpoonup u(t)$ in L^2 for $\forall t \in I$ fixed (representatives).

- *Proof:* $u_n(t) \rightharpoonup u(t)$ in $W = \left\{u(t) \in L^2\left(I;W_0^{1,2}\right),\ \frac{d}{dt}u(t) \in L\left(I;W^{-1,2}\right)\right\}$, but the mapping $u(\cdot) \mapsto u(t)$, is continuous and linear for $t \in I$ fixed as $W \to L^2$ by Th. 1.12. The proof is completed since the weak convergence is preserved by continuous mappings, see "Properties of weak convergence" in appendices.

Step 3: passage to the limit.

$$\begin{array}{lcl}\frac{d}{dt}u_n(t) + A(u_n(t)) + f(u_n(t)) & = & h(t) \text{ in } L^2\left(I;W^{-1,2}\right)\\ \quad\downharpoonleft \qquad\qquad \downharpoonleft \qquad\qquad \downharpoonleft & & \\ \frac{d}{dt}u(t) \qquad + \alpha(t) \quad + f(u(t)) & = & h(t)\end{array}$$

- easy to show: $f(u_n(t)) \to f(u(t))$ in $L^2\left(I;L^2\right)$, hence also $\rightharpoonup$ in $L^2\left(I;W^{-1,2}\right)$ (ex. 3).
- observe: $A(u_n(t))$ is bounded in $L^2\left(I;W^{-1,2}\right)$, hence $A(u_n) \rightharpoonup \alpha(t)$ in $L^2\left(I;W^{-1,2}\right)$ for some $\alpha(t)$ after taking a subsequence.

It remains to show that $A(u(t)) = \alpha(t)$ for a. a. $t \in I$.

Apply L. 2.3 (Minty's trick): , $X = L^2\left(I,W_0^{1,2}\right)$,$X^* = L^2\left(I,W^{-1,2}\right)$, $\tilde{A} : X \to X*$: $u(t) \mapsto A(u(t))$. easy to see: $\tilde{A}$ monotone and hemicontinuous (in fact continuous, HW3). Need to show: (This completes the proof.)

$$\limsup_{n\to\infty} \underbrace{\int_I \langle A(u_n)t), u_n(t)\rangle_{W^{-1,2},W_0^{1,2}}\,dt}_{\langle \tilde{A}(u_n),u_n\rangle_{X^*,X}} \leq \int_I \langle \alpha(t), u(t)\rangle_{W^{-1,2},W_0^{1,}}\,dt \qquad (*)$$

By step 1: $\frac{d}{dt}\|u_n\|_2^2 + \langle A(u_n(t)), u_n(t)\rangle + (f(u_n(t)), u_n(t)) = \langle h(t), u_n(t)\rangle \,/\, \int_0^T dt$, rearange:

$$\int_I \langle A(u_n(t)), u_n(t)\rangle\,dt = \int_I \langle h(t), u_n(t)\rangle\,dt - \int_I (f(u_n(t)), u_n(t))\,dt + \frac{1}{2}\|u_n(0)\|^2 - \frac{1}{2}\|u_n(T)\|^2$$

$$\limsup_{n\to\infty}\int_I \langle A(u_n(t)), u_n(t)\rangle\, dt = \limsup_{n\to\infty}(J_{1n}+J_{2n}+J_{3n}+J_{4n}) \le \sum_{k=1}^{4}\limsup_{n\to\infty} J_{kn}$$

$$\limsup_{n\to\infty} J_{1n} = \lim_{n\to\infty}\int_I \langle h(t), u_n(t)\rangle\, dt = \int_I \langle h(t), u(t)\rangle\, dt \text{ as } u_n \rightharpoonup u, \text{ in } L^2\left(I, W_0^{1,2}\right)$$

$$\limsup_{n\to\infty} J_{2n} = \lim_{n\to\infty}\int_I (f(u_n(t)), u_n(t))\, dt = \int_I \langle \iota f(u(t)), u(t)\rangle\, dt \text{ as } u_n \to u, \text{ in } L^2\left(I, L^2\right)$$

$$\limsup_{n\to\infty} J_{3n} = \lim_{n\to\infty}\frac{1}{2}\|u_n(0)\|_2^2 = \|u(0)\|_2^2$$

$$\limsup_{n\to\infty} J_{4n} = \limsup_{n\to\infty}\left(-\frac{1}{2}\|u_n(T)\|_2^2\right) = -\liminf_{n\to\infty}\left(\frac{1}{2}\|u_n(T)\|_2^2\right) \le -\frac{1}{2}\|u(T)\|_2^2$$

because $u_n(T) \rightharpoonup u(T)$ in L^2 and the norm is weakly lower-semicontinuous, i. e. $\|u(T)\|_2 \le \liminf_{n\to\infty}\|u_n(T)\|_2$.
Finally: $\limsup_{n\to\infty}\int_I \langle A(u_n(t)), u_n(t)\rangle\, dt \le \int_I \langle h(t), u(t)\rangle\, dt + \int_I (f(u(t)), u(t))\, dt + \frac{1}{2}\|u(0)\|_2^2 - \frac{1}{2}\|u(T)\|_2^2$.
But test eq. for $u(t)$ (three underbraces above), apply $\langle \cdot, u(t)\rangle$:

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 + \langle \alpha(t), u(t)\rangle + (f(t), u(t)) = \langle h(t), u(t)\rangle \quad / \int_I dt$$

obtain:

$$\int_I \langle \alpha(t), u(t)\rangle\, dt = \int_I \langle h(t), u(t)\rangle\, dt - \int_I (f(u(t)), u(t))\, dt + \frac{1}{2}\|u(0)\|_2^2 - \frac{1}{2}\|u(T)\|_2^2$$

by comparing the RHS we get the $(*)$, which completes the proof.

Remark: $u(t)$ is w. s. $\iff \frac{d}{dt}\iota u(t) = \mathcal{F}(t, u(t))$, where $\mathcal{F}(t,u): I\times W_0^{1,2} \to W^{-1,2}$, where

$$\mathcal{F}(t,u) = -\mathcal{A}(u) - \iota f(u) + h(t) \iff$$

$$\frac{d}{dt}(u(t), v) = \mathcal{F}\langle t, u(t)\rangle \text{ for } \forall v \in W_0^{1,2} \text{ fixed}$$

easy to show $\iff \frac{d}{dt}(u(t), w_j) = \langle \mathcal{F}(t, u(t)), w_j\rangle$, where $w_1, w_2, \ldots$ are dense in $W_0^{1,2}$ ... infinite (countable) system of ODEs. Good choice of basis w_j is useful, it makes things nice.

Recall [Weak formulation of laplacian in $W_0^{1,2}$"]: Eigenvalue problem for Dirichlet laplacian:

$$\begin{aligned} -\Delta u &= \lambda u \quad \text{in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

weak formulation: find $u \in W_0^{1,2}$ s. t. $(\nabla u, \nabla v) = \lambda(u,v)$ for $\forall v \in W_0^{1,2}$. We know: $\exists \lambda_j > 0, \lambda_j \to \infty, w_j \in W_0^{1,2}$ pairs of eigenvalue/eigenfunction s t. $\{w_j\}$ are complete ON basis of L^2 and also complete ON basis of $W_0^{1,2}$ ... Hilbert space with scalar product $((\cdot,\cdot)) = (\nabla u, \nabla v)$ ... (due to Poincaré it gives an equivalent norm on $W_0^{1,2}$)

Notation: P_n ... projection $L^2 \to \operatorname{span}\{w_1, \ldots, w_n\}$. Clearly $\|P_n\| = 1$, but also $\|P_n\| = 1$ with respect to $W_0^{1,2}$ norm, see ex. 4.2.

Theorem 2.3 [Existence of weak solutions of parabolic equation]. Let $u_0 \in L^2$, $h(t) \in L^2\left(I, W^{-1,2}\right)$. Then there is $u(t) \in L^2\left(I, W_0^{1,2}\right)$ a weak solution such that $u(0) = u_0$.

Proof: **Step 0:** approximating solutions $u^N(t) = \sum_{j=1}^N c_j^N(t) w_j$. (^{N} is index), w_j laplace eigenfunctions, $c_j^N(t) : I \to \mathbb{R}$ are AC s. t.

$$(P_N) \quad \frac{d}{dt}c_j^N(t) = \left\langle \mathcal{F}\left(t, u^N(t)\right), w_j \right\rangle \quad j = 1, \ldots, N$$

... system of N ODEs.
Note: $c_j^N(t) = \left(u^N(t), w_j\right)$ ... since w_j are ON in L^2. Hence $(P_N) \iff \frac{d}{dt}\left(u^N(t), w_j\right) = \left\langle \mathcal{F}\left(t, u^N(t), w_j\right)\right\rangle, j = 1 \ldots N \iff P_N\left(\frac{d}{dt}u^N(t) - \mathcal{F}(t, u(t))\right) = 0$. $c_j^N(t)$ solves (P_N) with init. cond. $c_j^N = (u_0, w_j), \forall j = 1, \ldots, N \iff u^N(0) = P_N u_0$ by ON of w_j, hence $u^N(0) \to u_0 = u(0)$ in L^2.
Easy to see: (HW3): RHS of (P_N) satisfies Carathéodory conditions $\implies \exists c_j^N(t) \in AC(I)$. Hence $u^N(t)$ are well-defined in I.
Step 1: a priori estimates (independent of N fixed)

$$(P_N) \quad \frac{d}{dt}c_j^N(t) + \left\langle \mathcal{A}\left(u^N(t), w_j\right)\right\rangle + \left(f\left(u^N(t)\right), w_j\right) = \langle h(t, w_j)\rangle \quad \Big/ \cdot c_j^N(t), \sum_{j=1}^N$$

- first term: $\sum_{j=1}^N \left(\frac{d}{dt}c_j^N\right) c_j^N = \frac{1}{2}\frac{d}{dt}\underbrace{\sum_{j=1}^N \left(c_j^N\right)^2}_{=\left\|u^N\right\|_2^2} = \frac{1}{2}\frac{d}{dt}\left\|u^N\right\|_2^2$, since $w_j \in L^2$ are ON, the sum is finite and $c_j^N \in AC$.
- second term: $\left\langle \mathcal{A}\left(u^N\right), u^N\right\rangle = \int_\Omega a\left(\nabla u^N\right)\nabla u^N \overset{(A1)}{\geq} \alpha_0 \left\|\nabla u^N\right\|_2^2 \geq c_1 \left\|u^N\right\|_{1,2}^2$ as in Th. 2.2.
- third term: $\left|\left(f\left(u^N(t)\right), u^N(t)\right)\right| \leq \ldots \leq c_2\left(1 + \left\|u^N(t)\right\|_2^2\right)$ as in Th. 2.2
- fourth (last) term: $\left|\left\langle h(t), u^N(t)\right\rangle\right| \leq \ldots \leq \frac{c_1}{2}\left\|u^N(t)\right\|_{1,2}^2 + \frac{1}{2c_1}\|h(t)\|_{1,2}^2$.

Finally: $\frac{d}{dt}\left\|u^N(t)\right\|_2^2 + c_1\left\|u^N(t)\right\|_{1,2}^2 \leq c_3 + c_2\left\|u^N(t)\right\|_2^2$.
By Gronwall: $u^N(t)$ bounded in $L^2\left(I, W_0^{1,2}\right)$ and $L^\infty\left(I, L^2\right)$.
We now need: $\frac{d}{dt}\iota u^N(t)$ bounded in $L^2\left(I, W^{-1,2}\right)$ ... use dual characterization of norm:

$$\left\|\frac{d}{dt}\iota u^N(t)\right\|_{-1,2} = \sup_{v \in W_0^{1,2}, \|v\|=1}\left\langle \frac{d}{dt}\iota u^N(t), v\right\rangle_{W^{-1,2}, W_0^{1,2}}$$

but for v fixed we have:

$$\begin{aligned}
\left\langle \frac{d}{dt}\iota u^N(t), v\right\rangle &\overset{P_N u^N = u^N \text{ due to the previous equality}}{=} \\
\left\langle \iota P_N \frac{d}{dt}u^N(t), v\right\rangle &= \left(P_N\frac{d}{dt}u^N(t), v\right) = \left(\frac{d}{dt}u^N(t), P_N v\right) = \left\langle \iota\frac{d}{dt}u^N(t), P_N v\right\rangle \\
&= \left\langle \mathcal{F}\left(t, u^N(t)\right), P_N v\right\rangle \\
&\leq \left\|\mathcal{F}\left(t, u^N(t)\right)\right\|_{-1,2}\underbrace{\|P_N v\|_{1,2}}_{\leq 1 \text{ because } \|v\|_{1,2} = 1 \text{ and } \|P_N\|_{\mathcal{L}\left(W_0^{1,2}\right)} = 1} \leq \left\|\mathcal{F}\left(t, u^N(t)\right)\right\|_{-1,2} \\
&\leq C\left(\left\|u^N(t)\right\|_{1,2} + \|h(t)\|_{-1,2}\right) \text{ by (HW 3)}
\end{aligned}$$

taking sup: $\left\|\frac{d}{dt}u^N(t)\right\|_{L^2(I, W^{-1,2})} \leq C$ independent of N.

Step 2: $\exists$ convergent subsequences (as in Thm. 2. 2) s. t.

$$\begin{aligned} u^N(t) &\rightharpoonup u(t) \text{ in } L^2\left(I, W_0^{1,2}\right) \\ \frac{d}{dt}u^N(t) &\rightharpoonup \frac{d}{dt}u(t) \text{ in } L^2\left(I, W^{-1,2}\right) \\ u^N(t) &\to u(t) \text{ in } L^2\left(I, L^2\right) \\ \mathcal{A}\left(u^N(t)\right) &\rightharpoonup \alpha(t) \text{ in } L^2\left(I, W^{-1,2}\right) \\ u^N &\rightharpoonup u(t) \text{ in } L^2 \text{ for any } t \in I \text{ fixed} \end{aligned}$$

Step 3: limit passage $(P_N) \to (P_I)$

$$(P_N) \quad \frac{d}{dt}\left(u^N, w_j\right) = \left\langle \mathcal{F}\left(t, u^N(t)\right), w_j\right\rangle \text{ for } j \leq N$$

$$\begin{aligned} \iff -\int_I \left(u^N(t), w_j\right)\varphi'(t)\,dt &+ \int_I \left\langle \mathcal{A}\left(u^N(t)\right), w_j\right\rangle \varphi(t)\,dt \\ &+ \int_I \left(f\left(u^N(t)\right), w_j\right)\varphi(t)\,dt = \int_I \left\langle h(t), w_j\right\rangle \varphi(t)\,dt \end{aligned}$$

for $\forall \varphi(t) \in D(I)$. Fix j, let $N \to \infty$.

$$\begin{aligned} \implies -\int_I \left(u(t), w_j\right)\varphi'(t)\,dt &+ \int_I \left\langle (\alpha(t)), w_j\right\rangle \varphi(t)\,dt \\ &+ \int_I \left(f(u(t)), w_j\right)\varphi(t)\,dt = \int_I \left\langle h(t), w_j\right\rangle \varphi(t)\,dt \end{aligned}$$

This holds for $\forall j$, by density of w_j in $W_0^{1,2}$ we can replace w_j by $v \in W_0^{1,2}$ arbitrary.

$$\implies \frac{d}{dt}\iota u(t) + \alpha(t) + \iota f(u(t)) = h(t) \text{ in } L^2\left(I, W^{-1,2}\right)$$

It remains to show, that $\mathcal{A}(u(t)) = \alpha(t)$, same argument as in Th. 2.2. This finishes the proof.

Remark: If $\mathcal{A}$ is linear (e. g. heat equation), then $\mathcal{A}(u_n(t)) \rightharpoonup \mathcal{A}(u(t))$ is automatic.

Recall: maximum principle: $\partial_t u - \Delta u = h(t)$, $u(0) \leq 0$, $h(t) \leq 0$ then $u(t) \leq 0$ for $\forall t \geq 0$. Classical argument: let (t_0, x_0) be the point of maximum, then $\partial_t u(t) = 0$, $\frac{\partial^2}{\partial x^2}u \leq 0$ at (t_0, x_0), then contradiction: $\partial_t u - \Delta u \geq 0$ up to some details.

Problem: it requires $u(t) \in C\left(I \times \bar{\Omega}\right) \cap C_t^1 \cap C_x^2$. Therefore this is not available for weak solution.
„weak argument“: $\frac{d}{dt}\int_\Omega u^+(t)\,dx \leq 0$. (Then $u^+ = 0 \implies u^+(t) = 0 \forall t \geq 0$. How to prove $\frac{d}{dt}\int_\Omega u^+(t)\,dx \leq 0$?)
Observe $u^+ = \psi(u)$ where $\psi(z) = \begin{cases} z & ; z \geq 0 \\ 0 & ; z < 0 \end{cases}$. Key point: ψ is convex. Convex funcions have nice behaviour with respect to diffusion.

$$\frac{d}{dt}\int_\Omega \psi(u)\,dx = \int_\Omega \psi'(u)\underbrace{\partial_t u}_{\Delta u \ldots}\,dx = \int_\Omega \psi'(u)\Delta u\,dx \overset{PP}{=} \int_\Omega \underbrace{-\psi''(u)}_{\text{infinite but with good sign}}(\nabla u \cdot \nabla u)\,dx$$

This is the basic idea, now let us do it rigorously.

Lemma 2.4 [Weak derivative of compound functions]. Let $\psi(z) : \mathbb{R} \to \mathbb{R}$, with ψ', ψ'' bounded.

(1) if $u \in W_0^{1,2}$, then $\psi(u) \in W_0^{1,2}$, $\nabla\psi(u) = \psi'(u)\nabla u$ in the weak sense and moreover $u \mapsto \psi(u)$ is continuous $W_0^{1,2} \to W_0^{1,2}$.
(2) if $u(t) \in L^2\left(I, W_0^{1,2}\right)$, $\frac{d}{dt}u(t) \in L^2\left(I, W^{-1,2}\right)$ then $\frac{d}{dt}\int_I \psi(u(t))\,dx = \left\langle \frac{d}{dt}u(t), \psi'(u(t))\right\rangle$ for a. e. $t \in I$.

Proof:

(1) given $u \in W_0^{1,2}$ there $\exists u_n \in C_c^\infty$, $u_n \to u$ in $W^{1,2}$, let $\varphi \in C_c^\infty$ be fixed.

$$\int_\Omega \psi(u_n) \frac{d\varphi}{dx_j} dx = -\int_\Omega \psi'(u_n) \nabla u_n \varphi dx \text{ by parts, all is smooth, zero boundary}$$

Take limit $n \to \infty$:

$$\int_\Omega \psi(u) \frac{\partial \varphi}{\partial x_j} dx = -\int_\Omega \psi'(u) \nabla u \varphi dx$$

hence $\nabla \psi(u) = \psi'(u) \nabla u$ weakly.
Continuity: mapping $u \mapsto \psi(u)$ is continous $W_0^{1,2} \to W_0^{1,2}$? We want to show, that if $u_n \to u$ in $W^{1,2}$, i. e., $u_n \to u$ in L^2 and $\nabla u_n \to \nabla u$ in L^2, then $\psi(u_n) \to \psi(u)$ in $W^{1,2}$, i. e., $\psi(u_n) \to \psi(u)$ in L^2 and $\psi'(u_n) \nabla u_n \to \psi'(u) \nabla u$. Routine argument (Lebesgue): ψ', ψ'' bounded, hence $|\psi(z)| \le C(1+|z|)$.

(2) Let first $u(t)$ be smooth, say $u(t) \in C^1(I, L^2)$, $\psi \in C^2$ Take t_1, $t_2 \in I$ fixed.

$$u(t_1) - u(t_2) = \int_{t_1}^{t_2} \frac{d}{dt} u(t)\, dx \text{ in } L^2$$

hence $u(t_2, x) - u(t_1, x) = \int_{t_1}^{t_2} \frac{d}{dt} u(t, x)\, dx$ for a. e. $x \in \Omega$.
We know: $\frac{d}{dt} u(t,x) \in C(I, L^2) \hookrightarrow L^1(I, L^1) = L^1(I \times \Omega) = L^1(\Omega \times I)$, Fubini ... see ex. 2.2.
Hence for a. e. $x \in \Omega$ the following is true: $\frac{d}{dt} u(t,x) \in L^1(I) \implies t \mapsto u(t) \in AC \implies t \mapsto \psi(u(t,x)) \in AC$ and $\frac{d}{dt} \psi(u(t,x)) = \psi'(u(t,x)) \frac{d}{dt} u(t,x)$. Finally:

$$\psi(u(t_2,x)) - \psi(u(t_1,x)) = \int_{t_1}^{t_2} \psi(u(t,x)) \frac{d}{dt} u(t,x)\, dt \quad \text{for a. e. } x \in \Omega$$

Integrate $\int_\Omega dx$:

$$\int_\Omega \psi(u(t_2))\, dx - \int_\Omega \psi(u(t_1))\, dx = \int_{t_1}^{t_2} \underbrace{\int_\Omega \psi'(u(t)) \frac{d}{dt} u(t)\, dx}_{\left(\frac{d}{dt} u(t), \psi'(u(t))\right)} dt = \int_{t_1}^{t_2} \left\langle \iota \frac{d}{dt} u(t), \psi'(u(t)) \right\rangle dt$$

If $u(t) \in L^2\left(I, W_0^{1,2}\right)$ is arbitrary, $\frac{d}{dt} u(t) \in L^2(I, W^{-1,2})$ ... $\exists u_n \in C^1\left(I, W_0^{1,2}\right)$ s. t. $u_n \to u(t)$ in $L^2\left(I, W_0^{1,2}\right)$, $\iota \frac{d}{dt} u_n(t) \to \frac{d}{dt} u(t)$ in $L^2(I, W^{-1,2})$ and $u_n \to u(t)$ in L^2 for all $t \in I$ (continuous representative).
By the above:

$$\int_\Omega \psi(u_n(t_2))\, dx - \int_\Omega \psi(u_n(t_1))\, dx = \int_{t_1}^{t_2} \left\langle \iota \frac{d}{dt} u_n(t), \psi'(u_n(t)) \right\rangle dt$$

let $n \to \infty$ (by Lebesgue: routine limit passage (ex. 4)):

$$\int_\Omega \psi(u(t_2))\, dx - \int_\Omega \psi(u(t_1))\, dx = \int_{t_1}^{t_2} \left\langle \frac{d}{dt} u(t), \psi'(u(t)) \right\rangle dt$$

Def.: [Nonnegative functions in spaces L^p, $W_0^{1,2}$, $W^{-1,2}$]. If $v \in L^2$ or $W_0^{1,2}$, then $v \ge 0$ (or $v \le 0$) means $v(x) \ge 0$ (or $v(x) \le 0$) for a. e. $x \in \Omega$.
If $h \in W^{-1,2}$ then $h \ge 0$ (or $h \le 0$) means $\langle h, v \rangle \ge 0$ (or $\langle h, v \rangle \le 0$) for all $v \in W_0^{1,2}$, $v \ge 0$.

Theorem 2.4 [Maximum principle for weak solution of parabolic equation]. Let $u(t) \in L^2\left(I, W_0^{1,2}\right)$ be w. s., let $u(0) \le 0$, $h(t) \le 0$ for a. e. $t \in I$, $f(\cdot) \ge 0$. Then $u(t) \le 0$ for a. e. $t \in I$.

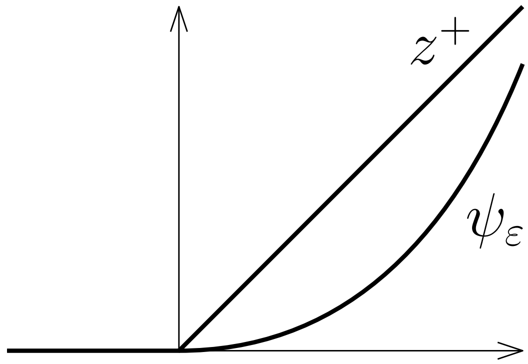

FIGURE 2.0.2. Graph of z^+ and ψ_ε.

Proof: take $\psi_\varepsilon : \mathbb{R} \to \mathbb{R}$ s. t. $0 \le \psi_\varepsilon \le z^+$, $\psi_\varepsilon(z) \nearrow z^+$, $\varepsilon \to 0^+$, ψ_ε', $\psi_\varepsilon'' \ge 0$ bounded.

$$\frac{d}{dt}\iota u(t) + \mathcal{A}(u(t)) + \iota f(u(t)) = h(t) \text{ in } W^{1,2}, \text{ take } \Big\langle \cdot, \underbrace{\psi_\varepsilon'(u(t))}_{\text{in } W_0^{1,2} \text{ by L. 2.4}} \Big\rangle$$

first term: $\left\langle \frac{d}{dt}\iota u(t), \psi_\varepsilon'(u(t)) \right\rangle = \frac{d}{dt}\int_\Omega \psi_\varepsilon(u(t))\,dt$ by L2.4
second term: $\langle \mathcal{A}(u(t)), \psi_\varepsilon'(u(t))\rangle = a(\nabla u, \nabla \psi_\varepsilon'(u)) \overset{L2.4}{=} (a(\nabla u), \psi_\varepsilon''(u)\nabla u) = \int_\Omega \underbrace{a(\nabla u)\cdot \nabla u}_{\ge 0 \text{ by } (A1)} \underbrace{\psi_\varepsilon''}_{\ge 0}\,dx \ge 0.$

third term$\langle \iota f(u), \psi_\varepsilon'(u)\rangle = \int_\Omega f(u)\psi_\varepsilon'(u) \ge 0$
fourth term:$\langle h(t), \psi_\varepsilon'(u)\rangle \le 0$
together:$\frac{d}{dt}\int_\Omega \psi_\varepsilon(u(t))\,dx \le 0$, take $\int_0^\tau dt$, $\tau \in I$: $\int_\Omega \psi_\varepsilon(u(\tau))\,dx \le \int_\Omega \psi_\varepsilon(u(0))\,dx$, take $\varepsilon \to 0^+$: $\int_\Omega u^+(\tau)\,dx \le \int_\Omega u^+(0)\,dx$, hence $u^+(0) = 0$ hence $u^+(\tau) = 0$ for all τ.

Theorem 2.5 [Strong solution of heat equation]. Let $u(t) \in L^2\left(I, W_0^{1,2}\right)$ be w. s. to the heat equation:

$$\frac{d}{dt}u - \Delta u + f(u) = h(t)$$

Let $u(0) \in W_0^{1,2}$, $h(t) \in L^2(I, L^2)$. Then $u(t) \in L^\infty\left(I, W_0^{1,2}\right) \cap L^2(I, W^{2,2})$, $\frac{d}{dt}u(t) \in L^2(I, L^2)$.

Remark: usually for w. s. $u(0) \in L^2$, $h(t) \in L^2(I, W^{-1,2})$ and $u(t) \in L^\infty(I, L^2) \cap L^2\left(I, W_0^{1,2}\right)$, $\frac{d}{dt}u(t) \in L^2\left(I, W_0^{-1,2}\right)$. Strong solution: one derivative better in spatial derivative

Proof: **Step 1:** (formal) ... multiply by $\frac{d}{dt}u$, $\int_\Omega dx$:

- first term: $\int_\Omega \frac{d}{dt}u \cdot \frac{d}{dt}u\,dx = \left\|\frac{d}{dt}u\right\|_2^2$
- second term: $-\int_\Omega \Delta u \frac{d}{dt}u\,dx = \int_\Omega \nabla u \cdot \nabla\left(\frac{d}{dt}u\right)dx = \frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2$
- third term: $\left|\left(f(u), \frac{d}{dt}u\right)\right| \le \|f(u)\|_2 \cdot \left\|\frac{d}{dt}u\right\|_2 \overset{\text{Young}}{\le} \|f(u)\|_2 + \frac{1}{4}\left\|\frac{d}{dt}u\right\|_2^2$ (Young: $ab \le \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2$, here $\varepsilon = 2$)
- fourth term: $\left|\left(h(t), \frac{d}{dt}u(t)\right)\right| \le \|h(t)\|_2 \cdot \left\|\frac{d}{dt}u\right\| \le \|h(t)\|_2^2 + \frac{1}{4}\left\|\frac{d}{dt}u\right\|_2^2$.

All together: $\frac{1}{2}\left\|\frac{d}{dt}u\right\|_2^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2 \le \|h(t)\|_2^2 + \|f(u)\|_2^2$ integrate $2\int_0^t dt$:

$$\int_0^t \left\|\frac{d}{dt}u(s)\right\|_2^2 ds + \|\nabla u(t)\|_2^2 \le \underbrace{2\int_0^t \|h(s)\|_2^2 + \|f(u(s))\|_2^2 ds}_{\le K} + \overbrace{\|\nabla u(0)\|_2^2}^{\text{ok, } u(0) \in W_0^{1,2}} \quad \forall t \in I$$

where K is independent of $t \in I$ because $h(t) \in L^2(I; L^2)$, $u(t) \in L^2(I; L^2)$ and f is lipschitz. Take $\sup_{t\in I}$: $\frac{d}{dt}u(t) \in L^2(I, L^2)$, $\|\nabla u(t)\|_2 \in L^\infty(I)$, by Poincaré ($\|\nabla u\| \approx \|u\|_{1,2}$ in $W_0^{1,2}$): $u \in L^\infty\left(I, W_0^{1,2}\right)$.

Step 2: rigorous proof: let $u^N(t) = \sum_{j=1}^N c_j^N(t) w_j$ be approximations from Thm. 2.3., i. e., $\frac{d}{dt}c_j^N + (\nabla u^N(t), \nabla w_j) + (f(u^N(t)), w_j) = (h(t), w_j)$ in $I, j = 1, \dots, N$, $c_j^N(0) = (u(0), w_j)$. Multiply by $\frac{d}{dt}c_j^N$, $\sum_{j=1}^N$:

- first term: note that $\frac{d}{dt}u^N(t) = \sum_{j=1}^N \frac{d}{dt}c_j^N(t) w_j$, hence $\sum_{j=1}^N \left(\frac{d}{dt}c_j^N\right)^2 = \left\|\frac{d}{dt}u^N\right\|_2^2$, since $w_j \in L^2$ are ON,
- second term: $\Big(\nabla u^N, \nabla \underbrace{\textstyle\sum_{j=1}^N \frac{d}{dt}c_j^N w_j}_{\frac{d}{dt}u^N}\Big) = \left(\nabla u^N, \frac{d}{dt}\nabla u^N\right) = \frac{1}{2}\frac{d}{dt}\left\|\nabla u^N\right\|_2^2$,
- third term: $\left|\left(f(u^N), \frac{d}{dt}u^N\right)\right| \overset{\text{C.S.}}{\leq} \left\|f(u^N)\right\|_2 \cdot \left\|\frac{d}{dt}u^N\right\|_2 \overset{\text{Young}}{\leq} \left\|f(u^N)\right\|_2^2 + \frac{1}{4}\left\|\frac{d}{dt}u^N\right\|_2^2$,
- fourth term: $\left|\left(h(t), \frac{d}{dt}(u^N)\right)\right| \leq \dots \leq \|h(t)\|_2^2 + \frac{1}{4}\left\|\frac{d}{dt}u^N\right\|_2^2$.

Together as before: $\frac{1}{2}\left\|\frac{d}{dt}u^N\right\|_2^2 + \frac{1}{2}\frac{d}{dt}\left\|\nabla u^N\right\|_2^2 \leq \|h(t)\|_2^2 + \left\|f(u^N)\right\|_2^2$, take $\int_I ds$:

$$\int_0^t \left\|\frac{d}{dt}u^N(s)\right\|_2^2 + \left\|\nabla u^N(t)\right\|_2^2 \leq 2\int_0^t \|h(s)\|_2^2 + \left\|f(u^N(s))\right\|_2^2 ds + \left\|\nabla u^N(0)\right\|_2^2$$

Observe: RHS $\leq K$ independent of $t \in I$ and N: beacuse $h(t) \in L^2(I, L^2)$, $u^N(t)$ is bounded in $L^\infty(I, L^2)$, $f(\cdot)$ lipschitz and $u^N(0) = P_N u(0)$, where $P_N : L^2 \to \operatorname{span}\{w_1, \dots, w_N\}$ but we also know $\|P_N u\|_{1,2} \leq \|u\|_{1,2}$. Hence $\left\|u^N(0)\right\|_{1,2}$ is bounded.

By argument of step 1: $u^N(t)$ bounded in $L^\infty\left(I, W_0^{1,2}\right)$ and $\frac{d}{dt}u^N(t)$ is bounded in $L^2(I, L^2)$. Therefore w. s. $u(t)$ constructed in Thm 2.3 belongs to these spaces. But we have uniqueness (Th. 2.1), therefore every solution has this regularity.

Step 3: $u(t) \in L^2(I, W^{2,2})$:

Recall elliptic regularity for laplacian: if $-\Delta u = F$ weakly (i. e. $(\nabla u, \nabla v) = (F, v)$ for $\forall v \in W_0^{1,2}$), where $F \in L^2$, then $u \in W^{2,2}$ and $\|u\|_{2,2} \leq C_R \|F\|_2$, C_R depends only on Ω.

$$\begin{aligned} -\int_I (u(t), v)\varphi'(t)\,dt \quad &+ \quad \int_I (\nabla u(t), \nabla v)\varphi(t)\,dt + \int_I (f(u(t)), v)\varphi(t)\,dt \\ &= \int_I (h(t), v)\varphi(t)\,dt \quad \forall v \in W_0^{1,2}\ \forall \varphi(t) \in D(I) \end{aligned}$$

but $\frac{d}{dt}u(t) = g(t) \in L^2(I, L^2)$ by step 2, hence first term$= \int_I (g(t), v)\varphi(t)\,dt$. Therefore:

$$\int_I (\nabla u(t), \nabla v)\varphi(t)\,dt = \int_I (F(t), v)\varphi(t)\,dt \quad \forall v, \varphi(t)$$

where $F(t) = h(t) - f(u(t)) - g(t)$. By standard argument: $(\nabla u(t), \nabla v) = (F(t), v)$ $\forall v \in W_0^{1,2}$ and for a. e. $t \in I$.

Now use elliptic regularity: $\|u(t)\|_{1,2} \leq C_R \|F(t)\|_2$. But $F(t) \in L^2(I, L^2)$ hence $u(t) \in L^2(I, W^{2,2})$.

Remarks:

(1) local regularity if $u(t)$ is w. s., $u(0) \in L^2$, $h(t) \in L^2(I, L^2)$, but $u(t) \in L^2\left(I, W_0^{1,2}\right) \implies u(\tau) \in W_0^{1,2}$ for a. e. $\tau \in (0, T) \implies u(t) \in L^\infty\left(\tau, T; W_0^{1,2}\right) \cap L^2\left(\tau, T, W_0^{1,2}\right)$, $\frac{d}{dt}u(t) \in L^2(\tau, T, L^2)$ for arbitrarily small $\tau > 0.$,

(2) even better data imply even better solutions.

CHAPTER 3

Hyperbolic second order equations

$$
\begin{aligned}
(H1)\quad \partial_{tt}u - \Delta u + \alpha\partial_t u + f(u) &= h(t), \quad t \in I, x \in \Omega \\
(H2)\quad u &= u_0, \quad t = 0, x \in \Omega \\
(H3)\quad \partial_t u &= u_{1,}, \quad t = 0, x \in \Omega \\
(H4)\quad u &= 0, \quad t \in I, x \in \partial\Omega
\end{aligned}
$$

Assumptions: (A) $\Omega \subset \mathbb{R}^n$ bounded domain, $\partial\Omega$ regular (lipschitz), $f(z): \mathbb{R} \to \mathbb{R}$, $|f(z_1) - f(z_2)| \leq l|z_1 - z_2|$, $\forall z_1, z_2 \in \mathbb{R}$, $\alpha \in \mathbb{R}$.

Def.: [Weak solution of hyperbolic equation]. Function $u(t) \in L^\infty\left(I, W_0^{1,2}\right)$ with $\frac{d}{dt}u \in L^\infty\left(I, L^2\right)$ is called weak solution of the problem $(H1)$, iff

$$
\frac{d^2}{dt^2}(u(t), v) + (\nabla u(t), \nabla v) + \alpha\left(\frac{d}{dt}u(t), v\right) + (f(u(t)), v) = (h(t), v)
$$

in the sense of distributions in $(0,T)$, for any $v \in W_0^{1,2}$ fixed.

Remarks:

(1) define $A: W_0^{1,2} \to W^{-1,2}$ by $\langle Au, v\rangle = (\nabla u, \nabla u)\,\forall v \in W_0^{1,2}$, $(A = -\Delta)$, $u(t)$ is w. s. $\overset{\text{as in L2.1}}{\Longleftrightarrow}$ $\frac{d^2}{dt^2}\iota u(t) + Au(t) + \alpha\iota\frac{d}{dt}u(t) + \iota f(u(t)) = \iota h(t)$ in $W^{-1,2}$ or by d'Alambert: $\frac{d}{dt}u(t) = v(t)$ in L^2,

$$
\frac{d}{dt}\iota v(t) + Au(t) + \alpha v(t) + \iota f(u(t)) = \iota h(t) \text{ in } W_0^{-1,2}
$$

(2) continuous representatives: clearly $u(t) \in W^{1,2}\left(I, L^2\right) \hookrightarrow C\left(I, L^2\right)$, $\frac{d}{dt}u(t) = v(t) \in W^{1,2}\left(I, W^{-1,2}\right) \hookrightarrow C\left(I, W^{-1,2}\right)$. We will show better continuity later.
I. e.: $u(t) \in L^\infty\left(I, W_0^{1,2}\right) \hookrightarrow L^2\left(I, L^2\right), \frac{d}{dt}u(t) \in L^2\left(I, L^2\right) \implies u(t) \in C^{0,\frac{1}{2}}\left(I, L^2\right)$ by L. 1.5 ex. 2.1, $\frac{d}{dt}u \in L^2\left(I, L^2\right) \overset{\iota}{\hookrightarrow} L^2\left(I, W^{-1,2}\right)$, $\frac{d^2}{dt^2}u(t) = -Au(t) - \alpha\frac{d}{dt}u(t) - \iota f(u(t)) + h(t) \in L^2\left(I, W^{-1,2}\right)$, therefore $\frac{d}{dt}u(t) \in C^{0,\frac{1}{2}}\left(I, W^{-1,2}\right)$.

However (see ex. 4.1): if $X \hookrightarrow Z$, and $u(t) \in L^\infty(I, X) \cap C(I, Z) \implies u(t) \in C\left(I, X^{\text{weak}}\right)$. Using this $u(t) \in C\left(I, \left(W_0^{1,2}\right)^{\text{weak}}\right)$, $\frac{d}{dt}u(t) \in C\left(I, \left(L^2\right)^{\text{weak}}\right)$.

We even have have $u(t) \in C\left(I, W_0^{1,2}\right)$ and $\frac{d}{dt}u(t) \in C\left(I, L^2\right)$ for $u(t)$ weak solution. See the last remark of this chapter.

Def.: $u(t) \in C\left(I, X^{\text{weak}}\right) \iff t \mapsto \langle f, u(t)\rangle$ is continuous for $\forall f \in X^*$ fixed.

Remark: proving uniqueness the same way as for solution of parabolic equation: let u_1, u_2 be weak solutions, $u_1(0) = u_2(0)$, $\partial_t u_1(0) = \partial_t u_2(0)$, set $u(t) = u_1 - u_2 \ldots$ is $u(t) = 0\ \forall t \in I$?
Subtract equations for u_1, u_2:

$$
\partial_{tt}u - \Delta u + \alpha\partial_t u + f(u_1) - f(u_2) = 0/\text{formally } \partial_t u, \int_\Omega dx
$$

- first term: $\int_\Omega \underbrace{\partial_{tt} u \partial_t u d}_{\frac{1}{2}\partial_t(\partial_t u)^2} x = \frac{1}{2}\frac{d}{dt}\|\partial_t u\|_2^2$,
- second term: $\int_\Omega \nabla u \cdot \nabla(\partial_t u)\, dx = \frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2$,
- third term: $\alpha\|\partial_t u\|_2^2$,
- fourth term: $|(f(u_1) - f(u_2), \partial_t u)| \le l\|u\|_2\|\partial_t u\|_2 \le \dots$

Together: $\frac{d}{dt}\left(\frac{1}{2}\|\partial_t u\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2\right) \le c\left(\|\partial_t u\|_2^2 + \|u\|_2^2\right)$, use Gronwall lemma.

Problem: we can not do this for w. s., since $\partial_{tt} u \in L^2\left(I, W^{-1,2}\right), \partial_t u \in L^2\left(I, L^2\right)$, and that is not enough for first and second term above.

Remark:

- Goal: existence and (more precisely "or") uniqueness of solution. The solution could be weak, strong, classical or on the other hand very weak, in measures, in distributions. The uniqueness is easy to prove for stronger solutions, whereas the existence is easier for weaker ones.
- Parabolic equation:

$$\frac{d}{dt}u + \mathcal{A}(u) + \dots \quad / \cdot u, \int_\Omega dx$$

$$\implies \frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \underbrace{\langle \mathcal{A}(u), u\rangle}_{\ge c_1\|u\|_{1,2}^2} + \dots \quad / \int_0^\tau dt$$

$$\|u(\tau)\|_2^2 + \int_0^\tau \|u(t)\|_{1,2}^2\, dt \le C = C(u_0, h(t), \Omega)$$

$\implies$ a priori estimates: $u(t) \in L^\infty\left(I, L^2\right) \cap L^2\left(I, W^{1,2}\right)$, $\frac{d}{dt}u(t) \in L^2\left(I, W^{-1,2}\right)$ (formal)
a priori estimates + approximation (Galerkin: $u^N(t) = \sum_{j=1}^N c_j^N(t)\, w_j$) $\implies$ existence of a solution in the class of solutions above. A priori estimates are valid ($u(t)$ is admissible test function) $\implies$ easy proof of uniquenes: (a priori estimates for difference of solutions $w = u - v$, Gronwall).
- hyperbolic equation:

$$\begin{aligned} \frac{d^2}{dt^2}u + Au + \alpha\frac{d}{dt}u + f(u) &= h(t,x) \\ u(0) &= u_0 \\ \frac{d}{dt}u(0) &= u_1 \end{aligned}$$

assumptions: $A = -\Delta$ (weak), $f(\cdot)$ globally lipschitz, $\alpha \in \mathbb{R}$, $h(t,x) \in L^2\left(I, L^2\right)$, $u_0 \in W_0^{1,2}$, $u_1 \in L^2$. Formal a priori estimates: use $\frac{d}{dt}u$ as a test functions, the first and second term yield: $\frac{d}{dt}E[u]$, where $E[u] = \frac{1}{2}\left(\left\|\frac{d}{dt}u\right\|_2^2 + \|\nabla u\|_2^2\right)$ is energy (sensible defined for physicists). Integrate $\int_0^\tau dt \implies E[u(\tau)] \le C = C(\text{data}) \implies u \in L^\infty\left(I, W_0^{1,2}\right)$, $\frac{d}{dt}u(t) \in L^\infty\left(I, L^2\right)$. Problem: this can not be done, since $\frac{d^2}{dt^2}u(t) \in L^\infty\left(I, W^{-1,2}\right)$ is not admissible test function.

Theorem 3.1 [Uniqueness of weak solution of hyperbolic equation]. The weak solution is uniquely defined by the initial conditions.

Proof: let $u(t)$, $v(t)$ be weak solutions, $u(0) = v(0)$, $\frac{d}{dt}u(0) = \frac{d}{dt}v(0)$, denote $w(t) = u(t) - v(t)$. $w(t)$ satisfies:

$$\frac{d^2}{dt^2}\iota w + Aw(t) + \alpha\iota\frac{d}{dt}w + \iota(f(u) - f(v)) = 0$$

(weakly in I: equation in $W^{-1,2}$, $w(0) = 0$, $\frac{d}{dt}w(0) = 0$)

Trick: $\int_0^t ds$: four terms:

(1) $\int_0^t \frac{d^2}{dt^2}\iota w ds = \left[\frac{d}{dt}\iota w\right]_0^t = \frac{d}{dt}\iota w(t)$

(2) $\int_0^t Aw(s)\,ds = A\underbrace{\int_0^t w(s)\,ds}_{\text{denote W}(t)\in W_0^{1,2}} = AW(t) \in W^{-1,2}$

(3) $\alpha\int_0^t \frac{d}{dt}\iota w(s)\,ds = \alpha\left[\iota w(s)\right]_0^t$

(4) $\iota\int_0^t f(u(s)) - f(v(s))\,ds$

Together:

$$\frac{d}{dt}\iota w + AW + \alpha W + \iota\int_0^t f(u(s)) - f(v(s))\,ds = 0 \quad \text{weakly in } I \text{ with values in } W^{-1,2},\ \langle\cdot,\underbrace{w}_{\in W_0^{1,2}}\rangle$$

(1) $\left\langle \frac{d}{dt}\iota w, w\right\rangle = \left(\frac{d}{dt}w, w\right) = \frac{1}{2}\frac{d}{dt}\|w\|_2^2$
Remark: special case of Th. 1.12: $u(t)\in L^2(I,H)$, $\frac{d}{dt}u \in L^2(I,H) \implies \frac{d}{dt}\|u(t)\|_H^2 = 2\left(\frac{d}{dt}u(t), u(t)\right)_H$ weakly. Apply for $H = L^2$.

(2) Apply previous remark for $H = W_0^{1,2}$:

$$\langle AW, w\rangle = (\nabla W,\nabla w) = \underbrace{\left(\left(W, \overbrace{w}^{=\frac{d}{dt}W}\right)\right)}_{\text{scalar product in } W_0^{1,2}} \stackrel{\text{remark}}{=} \frac{1}{2}\frac{d}{dt}\underbrace{\|W\|_{1,2}^2}_{\text{norm induced by } ((\cdot,\cdot))} = \frac{1}{2}\frac{d}{dt}\|\nabla W\|_2^2$$

(3) $|\alpha\langle\iota w, w\rangle| = |\alpha(w,w)| \le |\alpha|\,\|w\|_2^2$

(4)

$$\begin{aligned}\left|\left\langle \iota\int_0^t f(u(s)) - f(v(s))\,ds, w(t)\right\rangle\right| &= \left|\int_0^t (f(u(s)) - f(v(s))\,ds, w(t))\right| \\ &\le \int_0^t \underbrace{\|f(u(s)) - f(v(s))\|_2}_{\le \ell\|w(s)\|_2}\cdot\|w(t)\|\,ds \\ &\stackrel{f\text{ lipsch.}}{\le} \ell\int_0^t \|w(s)\|_2\|w(t)\|\,ds \\ &\stackrel{\text{Young}}{\le} \int_0^t \frac{\ell^2}{2}\|w(s)\|\,ds + \frac{t}{2}\|w(t)\|_2^2\end{aligned}$$

Together: $\frac{1}{2}\frac{d}{dt}\Big(\underbrace{\|w(t)\|_2^2 + \|\nabla W(t)\|_2^2}_{\text{denote by } Y(t)}\Big) \le \underbrace{\left(|\alpha| + \frac{t}{2}\right)}_{\le c_1}\underbrace{\|w(t)\|_2^2}_{\le Y(t)} + \frac{l^2}{2}\int_0^t\|w(s)\|_2^2\,ds$, $/\int_0^\tau dt$, we know: $Y(0) = 0$ due to the initial conditions:

$$Y(\tau) \le c_1\int_0^\tau Y(t)\,ds + \frac{l^2}{2}\underbrace{\int_0^\tau\int_0^t \|w(s)\|_2^2\,dsdt}_{\le c\int_0^\tau \|w(s)\|_2^2\,ds\ \dots\ \text{Fubini}}$$

So $Y(\tau) \le c_3\int_0^\tau Y(s)\,ds$ for $\forall\tau\in I$, Gronwall: $Y(\tau)\le 0$ pro $\forall\tau\in I$, i. e. $w(t) = 0$ a. e. in I. The proof is done. The trick works due to linearity of all problematic terms and the integration which increases regularity.

Lemma 3.1 [Testing hyperbolic equation by the derivative]. Let $u(t) \in L^2\left(I, W_0^{1,2}\right)$, $\frac{d}{dt}u(t) \in L^2\left(I, L^2\right)$, let $\frac{d^2}{dt^2}\iota u + Au = \iota H(t)$ weakly in I, where $H(t)\in L^2\left(I, L^2\right)$. Then $t\mapsto E[u(t)]$ is weakly differentiable and $\frac{d}{dt}E[u(t)] = \left(H(t), \frac{d}{dt}u(t)\right)$ for a. e. $t\in I$.

Proof: molifying by convolution: $\psi_n(t) = n\psi_0(nt)$, $\psi_0(t) \ldots$ convolution (smooth) molifier. Define $u_n(t) = u * \psi_n(t) \in C^\infty\left(J, W_0^{1,2}\right)$, where $J = (\delta, T-\delta)$, $\delta > 0$ fixed, $\frac{1}{n} < \delta$, here $u * \psi_n(t) = \int_{\mathbb{R}} u(s)\psi_n(t-s)\,ds = \int_{\mathbb{R}} u(t-s)\psi_n(t)\,ds$ makes sense.
We know: $u_n''(t) = u * \psi_n''(t) = \int_{\mathbb{R}} u(s)\psi_n''(t-s)\,ds$.
The equation (weakly): $\int_I \iota u(s)\varphi''(s)\,ds + \int_I Au(s)\varphi(s)\,ds = \int_I \iota H(s)\varphi(s)\,ds$ for $\forall\varphi(s) \in C_c^\infty(I)$ choose $\varphi(s) = \psi_n(t-s)$, $t \in J$ fixed, i. e..:

$$\langle \cdot, u_n'(t)\rangle \,/\quad \iota u_n''(t) + Au_n(t) = \iota H_n(t) \quad \text{in } J$$

(holds classically)

where $H_n(t) = H(t) * \psi_n(t)$. Thus we get:

$$\begin{aligned}
\frac{d}{dt}E[u_n(t)] &= \left\langle \iota H_n(t), \frac{d}{dt}u_n(t)\right\rangle = \left(H_n(t), \frac{d}{dt}u_n(t)\right) \quad \Big/ \int_0^\tau dt \\
E[u_n(\tau)] &= E[u_n(0)] + \int_0^\tau \left(H_n(t), \frac{d}{dt}u_n(t)\right)dt \quad ; n \to \infty \\
E[u(\tau)] &= E[u(0)] + \int_0^\tau \left(H(t), \frac{d}{dt}u(t)\right)dt
\end{aligned}$$

since $u_n(t) \to u(t)$ in $W_0^{1,2}$, $\frac{d}{dt}u_n(t) \to \frac{d}{dt}u_n(t)$ v L^2 for a. e. $t \in I$; $H_n(t) \to H(t)$ v $L^2\left(I, L^2\right)$, by which the proof is done.

Corollary: uniqueness of weak solution (other proof): apply L. 3.1 to $w(t) = u(t) - v(t)$, where $u(t)$ and $v(t)$ are solutions. I. e. $\frac{d^2}{dt^2}\iota w + Aw = \iota \underbrace{(-\alpha w + f(v) - f(u))}_{\text{denote by } H(t) \in L^2(I,L^2)}$, we get: $\frac{d}{dt}E[u(t)] = \left(-w + (f(u) - f(v)), \frac{d}{dt}w\right) \leq cE[w]$ (Young, routine estimates). Gronwall: $E[w(\tau)] \leq E[w(0)]\,e^{c\tau}$ for $\forall\tau \in I$. Uniqueness, moreover, continuous dependence on initial condition.

Theorem 3.2: [Existence of weak solution for hyperbolic equation]. Let $u_0 \in W_0^{1,2}$, $u_1 \in L^2$, $h(t) \in L^2\left(I, L^2\right)$. Then there exists $u(t) \in L^\infty\left(I, W_0^{1,2}\right)$ such that $\frac{d}{dt}u(t) \in L^\infty\left(I, L^2\right)$ is weak solution of $(H1)$ with initial conditions $u(0) = u_0$, $\frac{d}{dt}u(0) = u_1$ in the sense of continuous representatives.

Proof: **Step 1:** Galerkin approximation: $u^N(t) = \sum_{j=1}^N c_j^N(t)\,w_j$, $w_j \in W_0^{1,2}$ are laplace eigenfunctions, $c_j^N(t): I \to \mathbb{R}$, solve the system of equations

$$\begin{aligned}
(H_N) \quad \frac{d^2}{dt^2}\left(\iota u^N(t), w_j\right) \quad &+ \quad \left\langle Au^N(t), w_j\right\rangle + \alpha\left(\frac{d}{dt}u^N(t), w_j\right) \\
+ \left(f(u_n(t)), w_j\right) \quad &= \left(h(t), w_j\right) \quad \text{v } I, \forall j = 1, \ldots, N
\end{aligned}$$

Initial conditions: $u^N(0) = P_N u_0$, $\frac{d}{dt}u^N(0) = P_N u_1$.
Observe: $\left(u^N(t), w_j\right) = c_j^N(t)$, $\langle Au_N(t), w_j\rangle = \left(\nabla u^N(t), \nabla w_j\right) = \lambda_j \underbrace{\left(u^N(t), w_j\right)}_{c_j^N(t)}$. Therefore

$$\begin{aligned}
(H_N) \quad \partial_{tt}c_j^N + \lambda_j c_j^N + \alpha\frac{d}{dt}c_j^N + \mathcal{F}_j\left(t, c_1^N, \ldots, c_N^N\right) &= 0 \\
c_j^N(0) &= (u_0, w_j) \\
\frac{d}{dt}c_j^N(0) &= (u_1, w_j)
\end{aligned}$$

where $\mathcal{F}_j\left(t, c_1^N, \ldots, c_N^N\right) = \left(f\left(\sum_{l=1}^N c_l^N w_l\right), w_j\right) - (h(t), w_j)$. I. e., we have a system of nonlinear equations. The nonlinearity of $\mathcal{F}$ satisfies Carathéodory conditions (lipschitz with respect to c_k^N, integrable in t) $\Longrightarrow$ therefore there exists exactly one solution $c_j^N(t), \frac{d}{dt}c_j^N \in AC(I)$.

Step 2: estimates independent of N. $(H_N)\cdot\frac{d}{dt}c_j^N$, $\sum_{j=1}^N$ (formally analogous to multiplication of equation by time derivative and integration $\int_\Omega dx$). We get five terms:

(1) $\left(\frac{d^2}{dt^2}u^N,\frac{d}{dt}u^N\right)=\frac{1}{2}\frac{d}{dt}\left\|\frac{d}{dt}u^N\right\|_2^2$
(2) $\left\langle Au^N,\frac{d}{dt}u^N\right\rangle=\left(\nabla u^N,\nabla\left(\frac{d}{dt}u^N\right)\right)=\frac{1}{2}\frac{d}{dt}\left\|\nabla u^N\right\|_2^2$. Together with the first term gives $\frac{d}{dt}E\left[u^N\right]$
(3) $\left|\alpha\left(\frac{d}{dt}u^N,\frac{d}{dt}u^N\right)\right|\le|\alpha|\cdot\left\|\frac{d}{dt}u^N\right\|_2^2$
(4)

$$\begin{aligned}\left|\left(f\left(u^N\right),\frac{d}{dt}u^N\right)\right| &\le \left\|f\left(u^N\right)\right\|_2\left\|\frac{d}{dt}u^N\right\|_2\overset{\text{Ex. 3.2}}{\le}c\left(1+\left\|u^N\right\|_2\right)\cdot\left\|\frac{d}{dt}u^N\right\|_2\\ &\overset{\text{Poincaré}}{\le} c_1\left(1+\left\|\nabla u^N\right\|_2\right)\cdot\left\|\frac{d}{dt}u^N\right\|_2\\ &\overset{\text{Young}}{\le} c_3\left(1+\underbrace{\left\|\nabla u^N\right\|_2^2+\left\|\frac{d}{dt}u^N\right\|_2^2}_{2E[u^N]}\right)\end{aligned}$$

(5) $\left|\left(h(t),\frac{d}{dt}u^N\right)\right|\le\|h(t)\|_2\left\|\frac{d}{dt}u^N\right\|_2\le\frac{1}{2}\|h(t)\|_2^2+\underbrace{\frac{1}{2}\left\|\frac{d}{dt}u^N\right\|_2^2}_{\le E[u^N]}$

Together:

$$\frac{d}{dt}E\left[u^N(t)\right]\le c_4E\left[u^N(t)\right]+\frac{1}{2}\|h(t)\|_2^2\quad\Big/\int_0^\tau dt$$

$$E\left[u^N(\tau)\right]\le\underbrace{E\left[u^N(0)\right]+\frac{1}{2}\int_o^\tau\|h(t)\|_2^2}_{\le K\text{ independent of }N\ (☼)}+c_4\int_0^\tau E\left[u^N(t)\right]dt,$$

where K is independent of N due to $h(t)\in L^2\left(I,L^2\right)$ and Gronwall: $E\left[u^N(t)\right]\le K$, $\forall N\in\mathbb{N},\forall\tau\in I$, i. e., $u^N(t)$ is bounded in $L^\infty\left(I,W_0^{1,2}\right)$, $\frac{d}{dt}u^N(t)$ bounded in $L^\infty\left(I,L^2\right)$. The estimate (☼) holds since $h(t)\in L^2\left(I,L^2\right)$ and $E\left[u^N(0)\right]=\frac{1}{2}\|\nabla P_Nu_0\|_2^2+\frac{1}{2}\|P_Nu_1\|_2^2\le\frac{1}{2}\|\nabla u_0\|_2^2+\frac{1}{2}\|u_1\|_2^2$.
We claim: $\iota\frac{d^2}{dt^2}u^N(t)$ is bounded in $L^2\left(I,W^{-1,2}\right)$. Use: $\left\|\iota\frac{d^2}{dt^2}u^N(t)\right\|_{-1,2}=\sup_{\substack{w\in W_0^{1,2}\\ \|w\|=1}}\left\langle\iota\frac{d^2}{dt^2}u^N(t),w\right\rangle$:

$$\begin{aligned}\left\langle\iota\frac{d^2}{dt^2}u^N(t),w\right\rangle &= \left(\frac{d^2}{dt^2}u^N(t),w\right)=\left(\frac{d^2}{dt^2}P_Nu^N(t),w\right)\\ &= \left(\frac{d^2}{dt^2}u^N(t),P_Nw\right)=\left\langle\iota\frac{d^2}{dt^2}u^N(t),P_Nw\right\rangle\end{aligned}$$

Therefore $\left|\left\langle\iota\frac{d^2}{dt^2}u^N,w\right\rangle\right|=\left|\left(\frac{d^2}{dt^2}u^N(t),P_Nw\right)\right|\overset{*}{=}$
Aside:

$$(H_N):\quad\left\langle\iota\frac{d^2}{dt^2}u^N(t)+Au^N(t)+\alpha\iota\frac{d}{dt}u^n(t)+\iota f\left(u^N(t)\right)-\iota h(t),\underbrace{w_j}_{\text{generate }P_N}\right\rangle=0\quad\forall j=1,\dots,N.$$

Therefore $\iff$ $P_N\left(\iota\frac{d^2}{dt^2}u^N(t)+Au^N(t)+\alpha\iota\frac{d}{dt}u^n(t)+\iota f\left(u^N(t)\right)-h(t)\right)=0$ and $\left\langle\iota\frac{d^2}{dt^2}u^N(t),P_Nw\right\rangle=\left\langle-Au^N+\alpha\iota\frac{d}{dt}u^N+\iota f\left(u^N\right)-\iota h(t),P_Nw\right\rangle$
$\overset{*}{=}\left|\left\langle\iota\frac{d^2}{dt^2}u,w\right\rangle\right|\le\left\|-Au^N(t)-\alpha\iota\frac{d}{dt}u^N(t)-\iota f\left(u^N(t)\right)+\iota h(t)\right\|_{-1,2}\underbrace{\|P_Nw\|_{1,2}}_{\le\|w\|_{1,2}=1}$.
Use the estimates $\left\|Au^N(t)\right\|_{-1,2}\le\left\|\nabla u^N(t)\right\|_2$, $\left\|\iota\frac{d}{dt}u^N(t)\right\|_{-1,2}\le c\left\|\frac{d}{dt}u^N(t)\right\|$.

Together: $\left\|\iota \frac{d^2}{dt^2} u^N(t)\right\|_{-1,2} \leq c\Big(1 + \underbrace{\left\|\nabla u^N(t)\right\|_2 + \left\|\frac{d}{dt} u^N(t)\right\|_2 + \left\|u^N(t)\right\|_2 + \|h(t)\|_2}_{\text{bounded in } L^2(I) \text{ independently of } N}\Big)$.

Step 3: passage to the limit: Eberlein–Šmuljan: $\exists u(t)$ t. ž. $u^N(t) \overset{*}{\rightharpoonup} u(t)$ weakly-* v $L^\infty\left(I, W_0^{1,2}\right)$, $\frac{d}{dt} u^N(t) \overset{*}{\rightharpoonup} \frac{d}{dt} u(t)$ weakly-* in $L^\infty\left(I, L^2\right)$, due to Aubin–Lions lemma $u^N(t) \to u(t)$ strongly in $L^2\left(I, L^2\right)$ and therefore $f\left(u^N(t)\right) \to f(u(t))$ strongly in $L^2\left(I, L^2\right)$ (see ex. 3.2).
$(H_N) \cdot \varphi(t) \in C_c^\infty(I)$, $\int_I dt$:
first term: $\left(\frac{d^2}{dt^2} u^N(t), w_j\right) \varphi(t) \ldots 2\times$by parts, we get:

$$\int_I \left(u^N(t), w_j\right) \varphi''(t)\, dt + \int_I \left(\nabla u^N(t), \nabla w_j\right) \varphi dt + \alpha \int_I \left(\frac{d}{dt} u^N(t), w_j\right) dt + \int_I \left(f\left(u^N\right), w_j\right) \varphi(t)\, dt = \int_I (h(t), w_j) \varphi(t)\, dt$$

. $u^N \to u$ due to the convergences above we can take the limit in every term, holds for $\forall w_j$ since w_j are dense it holds for arbitrary $w \in W_0^{1,2}$, i. e., $u(t)$ is a weak solution.
Step 4: the initial conditions are satisfied (for continuous representatives): let us consider the space $\mathcal{W} = \left\{u(t) \in L^2\left(I, L^2\right), \frac{d}{dt} u(t) \in L^2\left(I, L^2\right)\right\} = W^{1,2}\left(I, L^2\right)$. We know: $u^N(t) \rightharpoonup u(t)$ v $\mathcal{W}$. Also: the operator $\tau : \mathcal{W} \to L^2$, $u(\cdot) \mapsto \tilde{u}(\tau)$ is continuous (L. 1.5), i. e. $u^N(\tau) \rightharpoonup \tilde{u}(\tau)$ in L^2 for $\forall \tau \in I$. We also know that $u^N(0) = P_N u_0 \to u_0$ strongly in $N \to \infty$. Therefore $\tilde{u}(0) = u_0$. For $u_1 = \frac{d}{dt} \tilde{u}(0)$ in $W^{-1,2}$ analogously. Thus the theorem is proven.

Theorem 3.3. [Strong solution of hyperbolic equation]. Let $u(t)$ be a weak solution of $(H1)$, let $\Delta u_0 + h(0) - f(u(0)) \in L^2$, $u_1 \in W_0^{1,2}$, $\frac{d}{dt} h(t) \in L^2\left(I, L^2\right)$ and $f(z) : \mathbb{R} \to \mathbb{R}$ is smooth with bounded derivative, let $\partial\Omega \in C^2$. Then $u(t) \in L^\infty\left(I, W^{2,2}\right)$, $\frac{d}{dt} u(t) \in L^\infty\left(I, W_0^{1,2}\right)$ and $\frac{d^2}{dt^2} u(t) \in L^\infty\left(I, L^2\right)$.

Proof: (formally): $\frac{d^2}{dt^2} u + Au + \alpha \frac{d}{dt} u = h(t) - f(u(t)) \quad \Big/ \frac{d}{dt}$, denote $v = \frac{d}{dt} u$. We get: $\frac{d^2}{dt^2} v + Av + \alpha \frac{d}{dt} v = \underbrace{\frac{d}{dt} h(t) - f'(u(t))\, v(t)}_{\text{denote by } H(t) \in L^2(I, L^2)}$, $v(0) = \frac{d}{dt} u(0) = u_1 \in W_0^{1,2}$, $\frac{d}{dt} v(0) = \frac{d^2}{dt^2} u(0) \overset{\text{equation for } u(t)}{=} \Delta u(0) + h(0) - f(u(0)) - \alpha \frac{d}{dt} u(0) \in L^2$. Estimates (see step 2 in Th. 3.2) for $v(t) \implies v(t) \in L^\infty\left(I, W_0^{1,2}\right)$, $\frac{d}{dt} v(t) \in L^\infty\left(I, L^2\right)$. It remains to show $u(t) \in L^\infty(I, W_{2,2})$: we want $\|u(t)\|_{2,2} \in L^\infty(I) \ldots$ again by the equation: $-\Delta u(t) = \underbrace{h(t) - f(u(t)) - \alpha \frac{d}{dt} u(t) - \frac{d^2}{dt^2} u(t)}_{\text{denote by } F(t)}$. By previous estimates we know (for $h(t)$ due to L. 1.5) we know that $F(t) \in L^\infty\left(I, L^2\right)$. By elliptic regularity: $\|u(t)\|_{2,2} \leq c_R \|F(t)\|_2$, where c_R depends on $\partial\Omega$, we need C^2 regularity of boundary. More correct proof: for $u^N(t)$ suitable approximations (i. e., Galerkin).

Remarks:

- If $u(t)$ is a weak solution of $(H1)$ and it holds that $u(t) \in L^\infty\left(I, W^{2,2}\right)$, $\frac{d}{dt} u(t) \in L^\infty\left(I, W_0^{2,2}\right)$, then $u(t) \in W^{2,2}$, $\frac{d}{dt} u(t) \in W_0^{1,2}$ for **every** $t \in I$, moreover, we have weak continuity with respect to t (see ex. 4.1).
- For parabolic equation (for simplicity, equation of heat): let $\frac{d}{dt} u - \Delta u = 0$, $u(0) = u_0 \in L^2$ (and not better), $\partial\Omega$ smooth.

  We know that there exists a weak solution $u(t) \in L^2\left(I, W_0^{1,2}\right) \implies \exists t > 0$ (arbitrarily small) s. t. $u(t_0) \in W_0^{1,2}$. Let us use Th. 2.5: $u(t) \in L^\infty\left(t_0, T; W_0^{1,2}\right) \cap L^2\left(t_0, T; W^{2,2}\right)$. $\implies \exists t_1 > t_0$ (arbitrarily close) s. t. $u(t_1) \in W^{2,2}$. We can arbitrarily increase the regularity as long as the boundary is sufficiently smooth. Therefore the regularity of weak solution increases in time.
- Hyperbolic equation, for simplicity: let $\frac{d^2}{dt^2} u - \Delta u + \alpha \frac{d}{dt} u = 0$, $\partial\Omega$ be smooth, $u(0) \in W_0^{1,2}$, $\frac{d}{dt} u(0) \in L^2$ and not better.

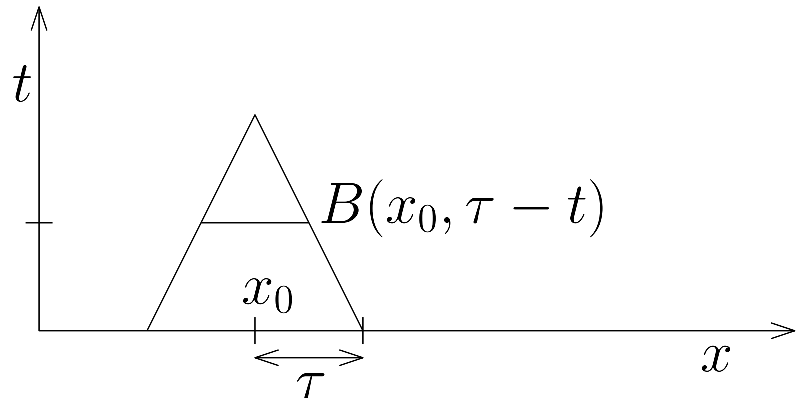

FIGURE 3.0.1. Wave principle illustration.

We claim: $\nexists \tau > 0$ s. t. $u(\tau) \in W^{2,2}$, $\frac{d}{dt}u(\tau) \in W_0^{1,2}$. Proof: time reversal in the wave equation: $u(t)$ is a weak solution in $[0,\tau]$ $\iff$ $\hat{u}(t) = u(\tau - t)$ is a weak solution in $[0,\tau]$, where $\alpha \leftarrow -\alpha$ (not significant for previous theory). Therefore the regularity can't be increased: if $u(\tau) \in W^{2,2}$, $\frac{d}{dt}u(\tau) \in W_0^{1,2} \implies \hat{u}(0) \in W^{2,2}$, $\frac{d}{dt}\hat{u}(0) \in W^{2,2}$, due to Th. 3.3 and following remark we have $\hat{u}(t) \in W^{2,2}$, $\frac{d}{dt}u(t) \in W_0^{1,2}$ for $\forall t$. Especially for $t = \tau$ we have a contradiction with non–smoothness of $u(0)$ and $\frac{d}{dt}u(0)$.

Notation:

- $B(x_0, r) = \{x \in \mathbb{R}^n, |x - x_0| < R\}$,
- $e = e(t,x) = \frac{1}{2}\left|\frac{d}{dt}u(t,x)\right|^2 + \frac{1}{2}|\nabla u(t,x)|^2$, where $u = u(t,x)$ is a weak solution. Remark: $E[u(t)] = \int_\Omega e(t,x) < dx$, in particular $e(t,x) \in L^1(I \times \Omega)$.

Theorem 3.4. [The wave principle]. Let $u(t)$ be a weak solution of the wave equation $\frac{d^2}{dt^2}u - \Delta u + \alpha\frac{d}{dt}u = 0$, where $\alpha \geq 0$. Let $x_0 \in \Omega$, $\tau \in I$ be such that $B(x_0,\tau) \subset \Omega$. Then $\int_{B(x_0,\tau-t)} e(t,x)\,dx \leq \int_{B(x_0,\tau)} e(0,x)\,dx$ for $\forall t \in [0,\tau]$.

Corollary [Finite speed of propagation]. $u(t)$ is a weak solution, $u(0) = 0$, $\frac{d}{dt}u(0) = 0$ in $B(x_0,\tau)$ $\implies$ $u = 0$ in $\bigcup_{t\in[0,\tau]} B(x_0, \tau - t)$.

Remark: analogous inequality holds for difference of solutions (linearity of the equation).

Proof: **Step 1:** let $u(t,x)$ be smooth. Define auxiliary function $p(t) = \int_{B(t)} e(t,x)\,dx$, where $B(t) = B(x_0, \tau - t)$. Auxiliary calculation (see "Time derivative of integral over time dependent domain" in Appendices): $p'(t) = \frac{d}{dt}\int_{B(t)} e(t,x)\,dx = \int_{B(t)} \partial_t e(t,x)\,dx - \int_{\partial B(t)} e(t,x)\,dS(x)$.
Goal: $p'(t) \leq 0$ (then we are done)

$$\partial_t e(t,x) = \partial_t\left(\frac{1}{2}(\partial_t u)^2 + \frac{1}{2}|\nabla u|^2\right) = \partial_{tt}u \cdot \partial_t u + \partial_t \nabla u \cdot \nabla u.$$

So we obtain:

$$p'(t) = \int_{B(t)} \partial_{tt}u \cdot \partial_t u + \underbrace{\partial_t \nabla u \cdot \nabla u}_{\text{by parts}}\,dx - \int_{\partial B(t)} e\,dS(x).$$

by parts (Gauss theorem): $\int_{\partial B(t)} (\partial_t u)\nabla u \cdot \underbrace{\nu}_{\text{outer normal}} dS(x) = \int_{B(t)} \nabla \cdot ((\partial_t u)\nabla u)\,dx = \int_{B(t)} \nabla\partial_t u \cdot \nabla u + (\partial_t u)\Delta u\,dx$.

Therefore $p'(t) = \int_{B(t)} \partial_{tt}u \cdot \partial_t u - \Delta u \cdot \partial_t u\,dx + \int_{\partial B(t)} -e + (\partial_t u)\nabla u \cdot \nu dS(x) = I_1 + I_2$.
ad I_1: by the equation we have: $\partial_{tt}u - \Delta u = -\alpha\partial_t u$, tj. $I_1 = \int_{B(t)} -\alpha(\partial_t u)^2\,dx \leq 0$ since $\alpha \geq 0$.

ad I_2: $\left|(\partial_t u)\nabla u\cdot\nu\right| \leq |\partial_t u|\,|\nabla u|\underbrace{|\nu|}_{=1} = |\partial_t u|\,|\nabla u| \overset{\text{Hölder}}{\leq} \frac{1}{2}|\partial_t u|^2+\frac{1}{2}|\nabla u|^2 = e$, therefore $I_2\leq 0$, since the integrand is ≤ 0. Therefore we are done for smooth solution.

Step 2: $u(t)$ is an arbitrary weak solution ... suitable approximation $u^n(t)$: sufficiently smooth so the first step is valid and at the same time $u^n(t)\to u(t)$ sufficiently strongly so the claim is preserved. Use Galerkin approximation $u^N(t)$ from Th. 3.2.

Observe: $u^N(t)=\sum_{j=1}^N c_j^N(t)\,w_j$ solves the equation: $(H_N)\quad \frac{d^2}{dt^2}c_j^N+\lambda_j c_j^N+\alpha\frac{d}{dt}c_j^N=0$, i. e., $c_k^N\in C^\infty(I)$ since it solves the ODE with constant coefficients. Moreover, $w_j\in C^\infty(\Omega)$ due to local regularity of laplacian: $-\Delta w_j=\lambda_j w_j$ ($PS\in L^2 \implies LS\in W^{2,2}\implies PS\in W^{2,2}$ etc.). Moreover, it holds that: $\frac{d^2}{dt^2}u^N-\Delta u^N+\alpha\frac{d}{dt}u^N=0$, i. e., the approximation exactly solves the equation. (Moreover, $c_j^N(t)\,w_j$ exactly satisfies the equation for $\forall j$ fixed.) Therefore the conclusion of theorem holds for u^N.

Passage to the limit: we know that $\nabla u^N(t)\rightharpoonup\nabla u(t)$, $\frac{d}{dt}u^N(t)\rightharpoonup\frac{d}{dt}u(t)$ in L^2, moreover $\nabla u^N(0)\to\nabla u(0)$, $\frac{d}{dt}u^N(0)\to\frac{d}{dt}u(0)$ v L^2. $t\in[0,\tau]$ is fixed, by weak lower semicontinuity we have:

$$\int_{B(t)}\frac{1}{2}\left|\frac{d}{dt}u(t,x)\right|^2+\frac{1}{2}|\nabla u(t,x)|^2\,dx\leq\liminf_{N\to\infty}\int_{B(t)}\frac{1}{2}\left|\partial_t u^N(t,x)\right|^2+\frac{1}{2}\left|\nabla u^N(t,x)\right|^2\,dx$$

$$\overset{\text{Step 1 for }u^N(t)}{\leq}\liminf_{N\to\infty}\int_{B(0)}\frac{1}{2}\left|\partial_t u^N(0,x)\right|+\frac{1}{2}\left|\nabla u^N(0,x)\right|^2\,dx$$

$$=\int_{B(0)}\frac{1}{2}\left|\frac{d}{dt}u(0,x)\right|^2+\frac{1}{2}|\nabla u(0,x)|^2\,dx$$

.

Remark: corollary of Th. 3.3: strong continuity of weak solutions: let $u(t)$ be a weak solution of $(H1)$, $u_0\in W_0^{1,2}$, $u_1\in L^2$, let $\partial\Omega\in C^2$, where $f(z)$ is smooth with bounded derivative, $\frac{d}{dt}h(t)\in L^2\left(I,L^2\right)$. Then we claim $u(t)\in C\left(I,W_0^{1,2}\right)$, $\frac{d}{dt}u(t)\in C\left(I,L^2\right)$.

Proof: choose $u_0^n\in W^{2,2}$, $u_1^n\in W_0^{1,2}$ s. t. $u_0^n\to u_0$ strongly in $W_0^{1,2}$ and $u_1^n\to u_1$ strongly in L^2. Let $u^n(t)$ be a strong solution for more smooth initial conditions u_1^n, u_0^n. Therefore $u^n(t)$ is at the same time a strong solution (Th. 3.3). At the same time it holds that $u^n(t)\rightharpoonup u(t)$ in the sense of Th. 3.3.

Trick: the equation for $u^n(t)-u^m(t)$, test by $\frac{d}{dt}(u^n(t)-u^m(t))$. (Standard estimates + Gronwall) ... we obtain: $E[u^m(t)-u^n(t)]\leq c_0\cdot E[u^m(0)-u^n(0)]$. Initial conditions $u^n(0)$, $\frac{d}{dt}u^n(0)$ are cauchy in $W_0^{1,2}$ resp. L^2, i. e., $u^n(t)$, $u^m(t)$ are cauchy in $C\left(I,W_0^{1,2}\right)$ resp. $C\left(I,L^2\right)$. By completeness & continuity of strong solution we obtain the conclusion.

CHAPTER 4

Semigroup theory

Up to now we considered evolution PDEs: $\frac{d}{dt}u - \Delta u + \cdots$. The rest of lectures is more close to functional analysis.

4.1. Homogeneous equation

Motivation: Let us have the equation $x' = ax$, where $x(0) = 1$ ... the solution is e^{at}, an exponential function. Generalization: $a \leftarrow A \in \mathbb{R}^{n\times n}$: $x' = Ax$, $x(0) = x_0$ the solution is $e^{tA}x_0$, a matrix exponential function. Goal: generalization to general Banach space, the study of equations of the type

$$\frac{d}{dt}x = Ax,\ x(0) = x_0,\ x \in X, \tag{4.1}$$

where X is a Banach space, $A : X \to X$ is a linear operator, e. g., $A = \Delta$. How to define a general exponential function e^{tA}? The power series is suitable only for bounded operators. Problem: Δ is unbounded operator, $\sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$ in general does not make sense. Remark: $-\Delta : W_0^{1,2} \to W^{-1,2}$ is bounded, but in different spaces.

Idea: A is "well unbounded" (i. e., bounded from above), then e^{tA} will be possible to define for $t > 0$.

Notation: [Unbounded operator].

- X ... Banach space with respect to $\|\cdot\|$.
- $\mathcal{L}(X) = \{L : X \to X$ is linear continuous operator$\}$ is a Banach space, $\|L\|_{\mathcal{L}(X)} = \sup_{\substack{x\in X \\ \|x\|=1}} \|Lx\|$,
- Unbounded operator is the couple $(A, D(A))$, where $D(A) \subset\subset X$ is a subspace (domain of definition of A), $A : D(A) \to X$ is linear.

Def.: [Semigroup, c_0-semigroup]. The function $S(t) : [0;\infty) \to \mathcal{L}(X)$ is called a semigroup, iff

(1) $S(0)$ is identity
(2) $S(t)\,S(s) = S(t+s)$, $\forall t, s \geq 0$

If moreover

(3) $S(t)\,x \to x$, $t \to 0^+$ for $\forall x \in X$ fixed, we call $S(t)$ a c_0-semigroup.

Remark:

- c_0-semigroup ... abstract exponential. Possible definitions of standard exponential: either a solution of $x' = ax$, $x(0) = 1$ or a power series $e^{at} = \sum_{n=0}^{\infty} \frac{(at)^n}{n!} = \lim_{n\to\infty}\left(1 + \frac{at}{n}\right)^n$, or a solution of functional equation: $f(x+y) = f(x)\,f(y)$ + continuity and $f(\cdot)$ is nonzero. Then „$S(t) = e^{ta}$“, c_0-semigroup is a suitable candidate for exponential.
- stronger assumption (3′) $\|S(t) - I\|_{\mathcal{L}(X)} \to 0$, $t \to 0^+$ (so called uniform continuity) implies $S(t) = e^{tA}$ for some linear continuous operator A, see ex. 5.1.

Lemma 4.1 [Exponential estimates, continuity in time of c_0-semigroup]. Let $S(t)$ be a c_0-semigroup in X. Then

(1) $\exists M \geq 1$, $\omega \geq 0$ s. t. $\|S(t)\|_{\mathcal{L}(X)} \leq M \cdot e^{\omega t}$ for $\forall t \geq 0$.
(2) $t \mapsto S(t)\,x$ is continuous $[0,\infty) \to X$ for $\forall x \in X$ fixed.

Proof:

(1) we claim: $\exists M \geq 1$, $\exists \delta > 0$ s. t. $\|S(t)\|_{\mathcal{L}(X)} \leq M$, $\forall t \in [0,\delta]$: by contradiction: if not, then $\exists t_n \to 0^+$ s. t. $\|S(t_n)\|_{\mathcal{L}(X)} \to +\infty$, but $S(t_n)\,x \to x$ for $\forall x \in X$ fixed due to the part (3) of semigroup definition and so $\|S(t_n)\,x\|$ is bounded. That is a contradiction to the principle of uniform boundednes, see functional analysis (a set of operators is bounded in operator norm iff $\|S(t_n)\,x\|$ is bounded for $\forall x$).
Set $\omega = \frac{1}{\delta}\ln M$, i. e.. $M = e^{\omega\delta}$, then for $t \geq 0$ arbitrary it holds that $t = n\delta + \varepsilon$, $\varepsilon \in [0,\delta)$, $n \in \mathbb{N}$. Then $\|S(t)\|_{\mathcal{L}(X)} = \big\|S\big(\underbrace{\delta + \delta + \cdots + \delta}_{n\times} + \varepsilon\big)\big\|_{\mathcal{L}(X)} = \|S(\delta)\cdots S(\delta)\,S(\varepsilon)\|_{\mathcal{L}(X)} \leq \|S(\delta)\|^n_{\mathcal{L}(X)}\,\|S(\varepsilon)\|_{\mathcal{L}(X)} \leq M \cdot \underbrace{M^n}_{=e^{\omega n\delta}} \leq M\cdot e^{\omega t}$.

(2) continuity: in 0^+ we have due to part (3) of definiton of semigroup. Continuity (from the right and from the left) in $t > 0$ remains:
Continuity from the right: $S(t+h)\,x = S(t)\underbrace{S(h)\,x}_{\to x} \to S(t)\,x$, $h \to 0^+$ due to the property (3), $S(t) \in \mathcal{L}(X)$.
Continuity from the left: (WLOG $h < t$) $S(t-h)\,x - \underbrace{S(t)\,x}_{S(t-h)S(h)x} = S(t-h)\,[x - S(h)\,x]$. Estimate:

$$\|S(t-h)\,x - S(t)\,x\| \leq \underbrace{\|S(t-h)\|_{\mathcal{L}(X)}}_{\leq Me^{\omega t},\text{ independent of } h \text{ due to the first part}} \overbrace{\|x - S(h)\,x\|}^{\to 0 \text{ due to (3)}},\ h \to 0^+.$$

Def.: [Generator of a semigroup]. An unbounded operator $(A, D(A))$ is called a generator of semigroup $S(t)$ iff

$$Ax = \lim_{h\to 0^+}\frac{1}{h}\left(S(h)\,x - x\right),\ D(A) = \left\{x \in X, \lim_{h\to 0^+}\frac{1}{h}\left(S(h)\,x - x\right) \text{ exists v } X\right\}.$$

Remark: it is easy to show that the operator defined by this formula is linear and $D(A) \subset X$ is a linear subspace.

Theorem 4.1 [Basic properties of a generator]. Let $(A, D(A))$ be a generator of $S(t)$, a c_0-semigroup in X. Then:

(1) $x \in D(A) \implies S(t)\,x \in D(A)$ for $\forall t \geq 0$,
(2) $x \in D(A) \implies AS(t)\,x = S(t)\,Ax = \frac{d}{dt}S(t)\,x$ for $\forall t \geq 0$ (in $t = 0$ only from the right),
(3) $x \in X$, $t \geq 0 \implies \int_0^t S(s)\,xds \in D(A)$, $A\left(\int_0^t S(s)\,xds\right) = S(t)\,x - x$.

Proof:

(1) $x \in D(A)$, $t \geq 0$ given, $\underbrace{\frac{1}{h}\big(\overbrace{S(h)\,S(t)}^{=S(h+t)=S(t)\,S(h)\text{due to (2)}}\,x - S(t)\,x\big)}_{(*)} \overset{?}{\to} y \implies S(t)\,x \in D(A)\,, AS(t)\,x = y$

$$(*) = \frac{1}{h}\left(S(t)\,S(h)\,x - S(t)\,x\right) = S(t)\underbrace{\left(\frac{1}{h}\left(S(h)\,x - x\right)\right)}_{\to Ax} \to S(t)\,Ax$$

(2) $x \in D(A)\,\ldots\ AS(t)\,x = S(t)\,Ax$ see part 1, $\frac{d}{dt}S(t)\,x = S(t)\,Ax$ from the right for $\forall t \geq 0$, see first part $\left(\frac{1}{h}\left(S(t+h)\,x - S(t)\,x\right) \to S(t)\,Ax\right)$, $h \to 0^+$.
From the left? $\frac{S(t-h)x - S(t)x}{-h} \to S(t)\,Ax$ as $h \to 0^+$ for $t > 0$ fixed? $\frac{S(t-h)x-S(t)x}{-h} = S(t-h)\left[\frac{x-S(h)x}{-h}\right] -$

$S(t)\,Ax = S(t-h)\left[\frac{x-S(h)x}{-h}\right] - S(t-h)\,S(h)\,Ax \overset{?}{\to} 0$: $\underbrace{S(t-h)}_{\text{L. 4.1, 1: bounded in } \|\cdot\|_{\mathcal{L}(X)}}\left\{\underbrace{\left[\frac{S(h)x-x}{h}\right]}_{\to Ax} - \underbrace{S(h)\,Ax}_{\to Ax \text{ due to (3)}}\right\} \to 0$ as in L. 4.1, 2.

(3) Denote $y = \int_0^t S(s)\,xds$, $x \in X$ for $t > 0$ fixed.

$$\begin{aligned}\frac{1}{h}\left(S(h)\,y - y\right) &= \frac{1}{h}\Bigg(\underbrace{S(h)\int_0^t S(s)\,xds}_{\int_0^t S(s+h)\,xds,\text{ subst. -- a shift by } h} - \int_0^t S(s)\,xds\Bigg)\\ &= \frac{1}{h}\left(\int_h^{t+h} S(s)\,xds - \int_0^t S(s)\,xds\right)\\ &= \frac{1}{h}\int_t^{t+h} S(s)\,xds - \frac{1}{h}\int_0^h S(s)\,xds \underbrace{\overset{h\to 0^+}{\to}}_{\text{derivative of continuous integrand (L. 4.1) w. r. t. upper bound}} S(t)\,x - \overbrace{S(0)\,x}^{=\text{x}}\end{aligned}$$

Therefore $y \in D(A)$, $Ay = S(t)\,x - x$, which was to be proven.

Remark: The theorem states:

(1) $D(A)$ is invariant w. r. t. $S(t)$.
(2) $S(t)$, A commute in v $D(A)$, moreover $t \mapsto S(t)\,x$ is a classical solution of $\frac{d}{dt}x = Ax$, $x(0) = x_0$, if $x \in D(A)$.

Def.: [Closed operator]. We say that an unbounded operator $(A, D(A))$ is closed, iff: $u_n \in D(A)$, $u_n \to u$, $Au_n \to v \implies u \in D(A)$ and $Au = v$.

Remark: it is easy to show that $(A, D(A))$ is closed $\iff$ $D(A)$ is complete (i. e., Banach) with respect to the norm $\|u\| + \|Au\|$, the so-called graph norm.

Remark: unbounded, but closed operators: natural property of derivative in different function spaces, examples:

(1) $\mathscr{X} = L^1(I, X)$, $A : u(t) \mapsto \frac{d}{dt}u(t)$, $D(A) = W^{1,1}(I, X)$ ... see chap. 1: statement (see ex. 2.1): $u_n(t) \in W^{1,1}(I, X)$, $u_n(t) \to u(t)$ v $L^1(I, X)$, $\frac{d}{dt}u_n(t) \to g(t)$ v $L^1(I, X) \implies u(t) \in W^{1,1}(I, X)$, $\frac{d}{dt}u(t) = g(t)$. This is equivalent to closedness of $(A, D(A))$.
(2) $X = C^1([0,1])$ ... theorem from analysis: $f_n(t) \in C^1([0,1])$, $f_n(t) \rightrightarrows f(t)$ v $[0,1]$, $\frac{d}{dt}f_n(t) \rightrightarrows \frac{d}{dt}g(t)$ v $[0,1] \implies f(t) \in C^1([0,1])$, $\frac{d}{dt}f(t) = g(t)$. That is equivalent to closedness of "$\frac{d}{dt}$" in $C([0,1]) = X$ with the definition domain $C^1([0,1])$.

Theorem 4.2 [Density and closedness of generator]. Let $(A, D(A))$ be a generator of a c_0-semigroup $S(t)$ in X. Then $D(A)$ is dense in X and $(A, D(A))$ is closed.

Proof: **Density** ... $x \in X$ given. $x = \lim_{h\to 0^+} \underbrace{\frac{1}{h}\int_0^h S(s)\,xds}_{\in D(A) \text{ dle V.4.1.3}}$ (continuity of integrand), i. e., we have elements from the definition domain which approximate given element.

Closedness ... $x_n \in D(A)$ given, $x_n \to x$, $Ax_n \to y \overset{?}{\implies} x \in D(A)$, $Ax = y$. Observe: $s \mapsto S(s)\,x_n$ is C^1 since $\frac{d}{dt}S(s)\,x_n = S(s)\,Ax_n$ due to Th.4.1, 2 and due to Newton-Leibnitz we have $S(h)\,x_n - S(0)\,x_n = \int_0^h \frac{d}{ds}S(s)\,x_n ds = \int_0^h S(s)\,Ax_n ds$. Take a limit $n \to \infty$. $LHS \to S(h)\,x - x$, RHS ... exchange of lim and $\int$: $Ax_n \to y$, therefore $\|S(s)\,Ax - S(s)\,y\| \le \underbrace{\|S(s)\|_{\mathcal{L}(X)}}_{\text{bounded independently of } s \in [0,h]} \|Ax_n - y\|$, uniform convergence. Therefore by the limit we obtain $\frac{1}{h}(S(h)\,x - x) = \frac{1}{h}\int_0^h S(s)\,yds$, take $h \to 0^+$. $RHS \to y$ (continuity of integrand), i. e., $LHS \to y$ or in other words $x \in D(A)$, $Ax = y$, which was to be proven.

Remark: Theorem 4.2, proof of closedness: $S(h)\,x_n - x_n = \int_0^h S(s)\,Ax_n ds,\ x_n \in D(A)$ is needed. In Th. 4.1, 3 we already have $S(h)\,x - x = A\left(\int_0^h S(s)\,xds\right),\ x \in X$. Wouldn't it be possible to shift A into the integral straight away?
Problem: does it hold that $A\left(\int_I f(s)\,ds\right) = \int_I Af(s)\,ds$? For continuous operators A it holds, see ex. 1.1. For closed operators A it is possible to prove, see ex. 5.3. It is not possible to use this argument above as we are just proving the closedness of generator A.

Remark:

- Th. 4.1:$(A, D(A))$ is a generator of a semigroup $S(t) \implies \forall x_0 \in D(A)$ is $x(t) = S(t)\,x_0$ a classical solution of (4.1).
- Key problem: $(A, D(A))$ given, $\overset{?}{\implies} \exists c_0$-sg. $S(t)$ s. t. A is a generator of $S(t)$.

Lemma 4.2 [Uniqueness of a semigroup]. Let $S(t), \tilde{S}(t)$ be c_0-semigroups which have the same generator. Then $S(t) = \tilde{S}(t)$ for $\forall t > 0$.

Proof: **Trick:** $y(t) = S(T-t)\,\widetilde{S}(t)\,x,\ x \in D(A)$, check $y(t) \in C([0,T], X),\ y'(t) = 0\ \forall t \in (0,T) \implies y(T) = \widetilde{S}(T)\,y(0) = S(T)\,x$. $D(A)$ is dense in X (Th. 4.2)

Def.: [Resolvent, resolvent set, spectrum]. Let $(A, D(A))$ be an unbounded operator. We define
resolvent set: $\rho(A) = \{\lambda; \lambda I - A \to X \text{ is one-to-one}\} \subset \mathbb{R}$ (generally can be considered a subset of $\mathbb{C}$),
resolvent: $R(\lambda, A) = (\lambda I - A)_{-1} : X \to D(A),\lambda \in \rho(A)$,
spectrum: $\sigma(A) = \{\lambda \in \mathbb{C}, \lambda I - A \text{ is not invertible}\}$. Equivalently $\sigma(A) = \mathbb{C}\backslash\rho(A)$.

Remark:

- $(A, D(A))$ is closed $\implies R(\lambda, A) \in \mathcal{L}(X)$, since A is closed $\iff D(A)$ is Banach with graph norm $\|x\| + \|Ax\|$. Moreover by the closedness is t $A : D(A) \to X$ continuous. Banach theorem on open mapping: inversion is continuous, i. e., $R(\lambda, A) : X \to D(A)$ is continuous.
- the following relations hold:

$$\begin{aligned} (i) \quad AR(\lambda, A)\,x &= \lambda R(\lambda, A)\,x - x\ \forall x \in X, \\ (ii) \quad R(\lambda, A)\,Ax &= \lambda R(\lambda, A)\,x - x\ \forall x \in D(A), \\ (iii) \quad R(\lambda, A)\,x - R(\mu A)\,x &= (\mu - \lambda)\,R(\lambda, A)\,R(\mu, A)\,x\ \forall x \in X, \end{aligned}$$

where (iii) is so-called resolvent identity.

Proof of (i): $AR(\lambda, A)\,x = [(A - \lambda I) + \lambda I]\,R(\lambda, A)\,x = \underbrace{-(\lambda I - A)\,R(\lambda, A)\,X}_{-x} + \lambda R(\lambda, A)\,x.$

The other relations are proven similarly, (i)-(ii) $\implies AR(\lambda, A)\,x = RA(\lambda, A)\,x,\ x \in D(A)$. Heuristics: $R(\lambda, A) = \frac{1}{\lambda - A}$.

Lemma 4.3 [Formula for the resolvent by Laplace transform]. Let $(A, D(A))$ be a generator of c_0-semigroup $S(t)$; let $\|S(t)\|_{\mathcal{L}(X)} \le Me^{\omega t}$. Then $\lambda \in \rho(A)$ for $\forall\lambda > \omega$ and the resolvent can be expressed as $R(\lambda, A)\,x = \int_0^\infty e^{-\lambda t} S(t)\,xdt,\ \|R(\lambda, A)\|_{\mathcal{L}(X)} \le \frac{M}{\lambda - \omega}$.

Proof: WLOG: $\omega = 0$, (see ex. 6.2) since c_0-sg. generated by $(A, D(A)) \iff \widetilde{S}(t) = e^{-\omega t} S(t)$ is c_0-sg. generated by $\left(\widetilde{A}, D\left(\widetilde{A}\right)\right)$ where $\widetilde{A} = A - \omega I,\ D\left(\widetilde{A}\right) = D(A)$. Moreover $R\left(\lambda, \widetilde{A}\right) = R(\lambda + \omega, A)$.
Therefore $\|S(t)\|_{\mathcal{L}(X)} \le M, \lambda > 0 \overset{?}{\implies} \lambda \in \rho(A)$. Denote $\widetilde{R}x = \int_0^\infty e^{-\lambda t} S(t)\,xdt$ (Laplace transform of semigroup $S(t)$), $x \in X,\ \lambda > 0$ fixed. Integral defined: integrand continuous (L. 4.1), $\|\text{integrand}\| \le e^{-\lambda t} M\,\|x\| \in L^1(0,\infty)$, $\left\|\widetilde{R}x\right\| \le \int_0^\infty e^{-\lambda t} M\,\|x\|\,dt = \frac{M}{\lambda}\,\|x\|$, i. e., $\widetilde{R} \in L(X)$, $\left\|\widetilde{R}\right\|_{\mathcal{L}(X)} \le \frac{M}{\lambda}$.
We will show that $\widetilde{R}x \in D(A)$:

$$\frac{1}{h}\,[S(h) - I]\,\widetilde{R}x = \frac{1}{h}\left[\int_0^\infty e^{-\lambda t}\,\underbrace{S(h)\,S(t)}_{S(h+t)}\,x - e^{-\lambda t} S(t)\,x\,dt\right],$$

Substution in first integral: $\int_h^\infty e^{-\lambda(t-h)} S(t)\,x$, $\pm \int_0^h e^{-\lambda(t-h)} S(t)\,x$, together:

$$= \frac{e^{\lambda h}-1}{h} \underbrace{\int_0^\infty e^{-\lambda t} S(t)\,x\,dt}_{\widetilde{R}x} - \frac{e^{\lambda h}}{h} \int_0^h e^{-\lambda t} S(t)\,x\,dt$$

Take $h \to 0^+$: $\to \lambda \widetilde{R}x - x$.
I. e., $\widetilde{R}x \in D(A)$, $A\widetilde{R}x = \lambda\widetilde{R}x - x$, $\forall x \in X$, in other words $(\lambda I - A)\widetilde{R}x = x$, i. e. $\lambda I - A : D(A) \to X$ is onto.
Is injective? Let $x \in D(A)$ be fixed,

$$A\widetilde{R}x = A\left(\int_0^\infty e^{-\lambda t} S(t)\,x\right) \overset{\text{Ex. 5.4, } A \text{ closed (Th. 4.2.)}}{=} \int_0^\infty A\left(e^{-\lambda t} S(t)\,x\right) dt$$

$$(\text{exchange op. and sg: Th. 4.1, 2}) = \int_0^\infty e^{-\lambda t} S(t)\,Ax dt = \widetilde{R}Ax$$

, i. e. $A\widetilde{R} = \widetilde{R}A$ in $D(A) \implies \widetilde{R}(\lambda I - A)\,x = \lambda\widetilde{R}x - \widetilde{R}AX \overset{A\widetilde{R}x = \lambda\widetilde{R}x - x}{=} x$, i. e., $\forall x \in D(A) : \widetilde{R}(\lambda I - A)\,x = x$ … $\lambda I - A$ is injective! I. e., $\widetilde{R} = R(\lambda, A)$, the proof is done.

Def.: [Semigroup of contractions]. We say that $S(t)$ is a semigroup of contractions, if $\|S(t)\|_{\mathcal{L}(X)} \le 1$, $\forall t \ge 0$

Theorem 4.3 [Hille–Yosida (for contractions)]. Let $(A, D(A))$ be an unbounded operator. Then it is equivalent:

(1) $\exists c_0$-semigroup of contractions, which is generated by $(A, D(A))$.
(2) $(A, D(A))$ is closed, $D(A)$ is dense in X, $\lambda \in \rho(A)$ for $\forall \lambda > 0$ and it holds that $\|R(\lambda, A)\|_{\mathcal{L}(X)} \le \frac{1}{\lambda}$.

Proof:

(1) $\implies$ (2) we already have proven: Th. 4.2 (density and closedness), L. 4.3 (resolvent, $M = 1$, $\omega = 0$).
(2) $\implies$ (1) Yosida's approximation: $A_n = nAR(n, A)$, $n \in \mathbb{N}$. Strategy: $S(t) = \lim_{n\to\infty} e^{tA_n} x$

Step 1: we claim that $A_n = n^2 R(n, A) - nI$, i. e., $A_n \in \mathcal{L}(X)$, $nR(n, A)\,x \to x$, $n \to \infty$ for $\forall x \in X$ fixed, $A_n x \to Ax$, $n \to \infty$, $\forall x \in D(A)$.

(i) $AR(n, A) = nR(n, A) - I$, $\big/ \cdot n \implies \underbrace{nAR(n, A)}_{A_n} = n^2 R(n, A) - nI \in \mathcal{L}(X)$, since $R(n, A) \in \mathcal{L}(X)$.

(ii) convergence principle (ex. 1.4, 1): $F_n : X \to X$ linear, $\|F_n\|_{\mathcal{L}(X)} \le C$, $F_n y \to y$ for $\forall y \in S$ fixed, S dense in $X \implies F_n x \to x$ for $\forall x \in X$.
Apply $F_n = nR(n, A)$, $\|F_n\|_{\mathcal{L}(X)} \le 1$ due to $\|R(\lambda, A)\|_{\mathcal{L}(X)} \le \frac{1}{\lambda}$ $\forall \lambda > 0$, $S = D(A)$ dense in X (assumption), $y \in D(A) : nR(n, A)\,y = y + \underbrace{\underbrace{R(n,A)}_{\|\cdot\|_{\mathcal{L}(X)} \le \frac{1}{n}} \underbrace{Ay}_{\text{fixed}}}_{AR(n,A)\,y} \to y$, $n \to \infty$.

(iii) $A_n x = nAR(n, A)\,x = nR(n, A)\,Ax \to Ax$ due to (ii)

- **Step 2:** approximation of semigroup $S_n(t) = e^{tA_n} = \sum_{k=0}^\infty \frac{t^k A_n^k}{k!} \in \mathcal{L}(X)$, since $A_n \in \mathcal{L}(X)$ due to step 1. At the same time we know: $A_n = \underbrace{n^2 R(n, A) - nI}_{\text{are commutative}} \implies e^{tA_n} = e^{tn^2 R(n,A) - tnI} = e^{-nt} \cdot \underbrace{e^{n^2 tR(n,A)}}_{(*)}$. $\|(*)\|_{\mathcal{L}(X)} = \left\|\sum_{k=0}^\infty \frac{(nt)^k (nR(n,A))^k}{k!}\right\|_{\mathcal{L}(X)} \le \sum_{k=0}^\infty \frac{(nt)^k}{k!} \underbrace{\|nR(n, A)\|_{\mathcal{L}(X)}^k}_{\le 1} \le e^{nt}$. I. e., we have $\|S_n(t)\|_{\mathcal{L}(X)} \le 1$ for $\forall n$, in other words $S_n(t)$ are contractions.

Step 3: existence of $\lim_{n\to\infty} S_n(t)\,x$, for $\forall x \in X$ fixed. For now, let $x \in D(A)$.

Trick: let t be fixed.

$$\begin{aligned} S_n(t)x - S_m(t)x &= [S_n(t-s)S_m(s)x]_{s=0}^{s=t} = \left[e^{(t-s)A_n}e^{sA_m}x\right]_{s=0}^{t} \\ &= \int_0^t \frac{d}{ds}\left[e^{(t-s)A_n}e^{sA_m}x\right]ds \\ &= \int_0^t -A_n e^{(t-s)A_n}e^{sA_m}x + e^{(t-s)A_n}A_m e^{sA_m}x ds = (\star) \end{aligned}$$

We know: $A_nA_m = A_mA_n \ldots$ is implied by expression of resolvent ($R(n,A)$, $R(m,A)$ are commutative).

$$(\star) = \int_0^t \underbrace{\underbrace{e^{(t-s)A_n}e^{sA_m}}_{S_n(t-s)S_m(s)}}_{\|\cdot\|_{\mathcal{L}(x)} \le 1}[A_mx - A_nx]\,ds$$

$$\Longrightarrow \|S_n(t)x - S_m(t)x\| \le t\,\|A_mx - A_nx\|$$

$x \in D(A) \implies A_nx,\ A_mx \to Ax,\ A_nx - A_mx \to 0 \implies \{S_n(t)x\}$ satisfy B. C. condition uniformly with respect to $t \in [0,T]$, which is bounded interval, i. e., there exists $S_n(t) \overset{loc}{\rightrightarrows} S(t)x$ defined as this uniform limit. The convergence principle (density of $D(A)$,$\|S_n(t)\|_{\mathcal{L}(X)} \le 1$: $S(t)$ is clearly a c_0-semigroup of contractions) $\implies \exists$ limit for $\forall x \in X$.

Step 4: $S(t)$ is generated by the original $(A, D(A))$. Denote the generator of $S(t)$ by $\left(\widetilde{A}, D\left(\widetilde{A}\right)\right)$, let $x \in D(A)$.

$$S_n(t)x - x = \int_0^t \frac{d}{ds}S_n(s)xds = \int_0^t A_nS_n(s)xds = \int_0^t \underbrace{S_n(s)}_{\rightrightarrows S(s)}\overbrace{A_nx}^{\to Ax}ds$$

For $n \to \infty$: $LHS \to S(t)x - x = \int_0^t S(s)A(x)\,ds$, i. e., $\frac{1}{h}(S(h)x - x) = \frac{1}{h}\int_0^t S(s)Axds \to Ax$, $h \to 0^+$, in other words $x \in D\left(\widetilde{A}\right)$, $\widetilde{A}x = Ax$, i. e., we have $D(A) \subset D\left(\widetilde{A}\right)$, $\widetilde{A}x = Ax$ for $x \in D(A)$.

Opposite implication: choose $\lambda > 0$ arbitrarily:

$$\lambda \in \underbrace{\rho(A)}_{\text{assumption}} \cap \overbrace{\rho\left(\widetilde{A}\right)}^{\text{L.4.3: } \|S(t)\|_{\mathcal{L}(X)} \le 1 = Me^{\omega t}, M=1, \omega=0}$$

and so $\lambda I - \widetilde{A} = \underbrace{\lambda I - A \text{ in D}(A)}_{\text{injective, onto}}$, therefore $RHS = \lambda I - \widetilde{A} : D\left(\widetilde{A}\right) \to X$ is onto, injective: $D(A) \subset D\left(\widetilde{A}\right), \lambda \in \rho\left(\widetilde{A}\right)$.

Remarks:

(1) Generalized Hille–Yosida theorem: $(A, D(A))$ generates c_0-semigroup, satisfying the estimate $\|S(t)\|_{\mathcal{L}(X)} \le Me^{\omega t} \iff$ is closed, densely defined and it holds that $\lambda \in \rho(A)$ for $\forall \lambda > \omega$ and $\|R^n(\lambda, A)\|_{\mathcal{L}(X)} \le \frac{M}{(\lambda-\omega)^n}$ $\forall n \in \mathbb{N}$.
Proof: reduce to Th. 4.3: we know $\omega \ne 0 \ldots$ $\widetilde{S} = e^{-\omega t}S(t)$, $M \ne 1 \ldots$ by equivalent norm in X.

(2) Lumer–Phillips theorem: $(A, D(A))$ generates c_0-semigroup of contractions $\iff$ it is closed and densely defined and $\|\lambda x - Ax\| \ge \lambda\|x\|$ $\forall x \in X$ for $\forall \lambda > 0$ and $\exists \lambda_0 > 0$ s. t. $\lambda_0 I - A : D(A) \mapsto X$ is onto.

Proof: reduce to Th. 4.3:

$$\|\underbrace{(\lambda I - A)\,x}_{y}\| \geq \lambda \|\underbrace{x}_{R(\lambda,A)\,y}\| \iff \|R(\lambda,A)\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda}.$$

(3) Heuristic remarks:
- key condition for generator of semigroup of contractions is in some sense "$A \leq 0$": $\|\lambda x - Ax\| \geq \lambda\|x\| \ldots \|x \underbrace{-\tfrac{A}{\lambda}}_{\approx \geq 0} x\| \geq \|x\|$
- in the proof of Th. 4.3 we have used the approximations $A_n x \to Ax$; $A_n x = n^2 R(n,A) - nI$, $nR(n,A)\,x \to x$. For quick check and as clue to remember: $R(n,A) = \frac{1}{n-A} \implies nR(n,A) = \frac{n}{n-A} = \frac{1}{1-\frac{A}{n}} \to 1$; $A_n = \frac{n^2}{n-A} - n = \frac{n^2-n(n-A)}{n-A} = \frac{nA}{n-A} = \frac{A}{1-\frac{A}{n}} \to A$.

Remark: (4.1) $\ldots$ $\frac{d}{dt}x = Ax$, $x(0) = x_0$ $\ldots$ if $S(t)$ is c_0-semigroup, generated by $A \implies S(t)\,x_0$ $\ldots$ we have reasonable concept of solution...

- $x_0 \in D(A)$ $\ldots$ even a classical solution (Th. 4.1, 2),
- $x_0 \in X \backslash D(A)$ $\ldots$ $S(t)\,x_0 = \lim_{n\to\infty} S(t)\,x_n$ where $x_n \in D(A)$, $x_n \to x_0$ uniformly in $[0,T]$.

4.2. Nonhomogeneous equation

$$\frac{d}{dt}u = Au + f(t), \; u(0) = u_0 \in X \tag{4.2}$$

where $f(t) : I \to X$ is integrable, $I = [0,T]$, $(A, D(A))$ is unbounded operator.

Def.: [Classical and strong solution of nonhomogeneous task]. A function $u(t)$ is called:

classical solution of (4.2)**:** if $u(t) \in C^1(I,X) \cap C(I,D(A))$ satisfies the equation (4.2) for $\forall t \in I$,
strong solution of (4.2)**:** if $u(t) \in W^{1,1}(I,X) \cap L^1(I,D(A))$ satisfies the equation (4.2) for a. e. $t \in I$.

Remarks.

(1) Classical solution is a strong one $\ldots$ classical derivative is a weak derivative, continuity implies integrability. $\exists u$ classical solution, neccessarily $f \in C(I,X)$.
(2) $u(t)$ is a strong solution $\iff$ $u(t) \in L^1(I,D(A))$ and $u(t) = u_0 + \int_0^t Au(s) + f(s)\,ds$ for a. e. $t \in I$. Proof: L 1.3, moreover RHS $\ldots$ AC representative of $u(t) \in W^{1,1}$.
(3) $u(t) \in C(I,D(A))$ means $t \mapsto u(t)$ is continuous with respect to the graph norm $D(A)$: $\|u\| + \|Au\|$. $(A,D(A))$ is closed $\iff D(A)$ is complete (Banach) in graph norm. It is easy to show that if $(A,D(A))$ is closed, then $u(t) \in C(I,D(A)) \iff u(t) \in C(I,X)$, $u(t) \in D(A)$ for $\forall t \in I$ and $Au(t) \in C(I,X)$.

Lemma 4.4 [Hille theorem on characterization of $L^1(I;D(A))$]. Let $(A,D(A))$ be closed. Then $u(t) \in L^1(I,D(A)) \iff u(t) \in L^1(I,X)$, $u(t) \in D(A)$ for a. e. t, $Au(t) \in L^1(I,X)$. Moreover in a such case, $\int_I u(t)\,dt \in D(A)$, $A\left(\int_I u(t)\,dt\right) = \int_I Au(t)\,dt$.

Proof: "$\implies$" and moreover $\ldots$ see ex. 5.3.
"$\impliedby$": observe: for $u_1(t) : I \to X_1$, $u_2(t) : I \to X_2$ integrable and X_1, X_2 Banach spaces is $(u_1(t), u_2(t)) : I \to X_1 \times X_2$ integrable, where we consider the norm $\|(u_1,u_2)\|_{X_1\times X_2} = \|u_1\|_{X_1} + \|u_2\|_{X_2}$ and $\int_I (u_1(t), u_2(t))\,dt = \left(\int_I u_1(t)\,dt, \int_I u_2(t)\,dt\right)$ $\ldots$ easy to prove by expansion of definitions.
Apply this observation to $u_1(t) = u(t)$, $u_2(t) = Au(t)$, $x_1 = x_2 = x$ $\ldots$ u_1,$u_2 : I \to D(A)$.

Def.: [Mild solution]. Let $(A,D(A))$ be a generator of c_0-semigroup $S(t)$. Then the function $u(t)$, satisfying $u(t) = S(t)\,u_0 + \int_0^t S(t-s)\,f(s)\,ds$, $\forall t \in I$ is called a mild solution of (4.2).

Remark: motivation: "variation of constans" since $S(t) = e^{tA}$. Integral is finite, therefore $u(t) \in C(I,X)$ $\ldots$ see next lemma, trivially $\exists!$ mild solution.

Lemma 4.5 [Properties of convolution of semigroup]
. Let $S(t)$ be a c_0-semigroup, let $v(t) = \int_0^t S(t-s) f(s)\, ds$. Then:

(1) $f(t) \in L^1(I,X) \implies v(t) \in C(I,X)$
(2) $f(t) \in C^{0,1}(I,X) \implies v(t) \in C^{0,1}(I,X)$
(3) $f(t) \in C^1(I,X) \implies v(t) \in C^1(I,X)$ and $v'(t) = S(t) f(0) + \int_0^t S(t-s) f'(s)\, ds\ \forall t \in I$.

Proof: $v(t) = \int_0^t S(t-s) f(s)\, ds$, $S(t-s) f(s) = g(s) \in L^1((0,t),X)$ (excercise: use measurability and integrability of f, $S(t-s) \in \mathcal{L}(X)$, $\|S(t)\|_{\mathcal{L}(X)} \leq c_0 \ldots$ L. 4.1, $t \in I = [0,T]) \ldots$ well defined for $f(t) \in L^1(I,X)$.

(1) continuity: let $t, t+h \in I$, WLOG $h > 0$.

$$\begin{aligned} v(t+h) - v(t) &= \int_0^{t+h} S(t+h-s) f(s)\, ds - \int_0^t S(t-s) f(s)\, ds \\ &= \int_0^h S(t+h-s) f(s)\, ds + \underbrace{\int_h^{t+h} S(t+h-s) f(s)\, ds}_{\text{subst. } s \to s+h} \\ &\quad - \int_0^t S(t-s) f(s)\, ds \\ &= \int_0^h S(t+h-s) f(s)\, ds + \int_0^t S(t-s) [f(s+h) - f(s)]\, ds \end{aligned}$$

$$\implies \|v(t+h) - v(t)\| \leq C_0 \int_0^h \|f(s)\|\, ds + C_0 \int_0^T \|f(s+h) - f(s)\|\, ds.$$

Set $f(t) = 0$ outside $I = [0,T]$, we know: $h \to 0$: $RHS \to 0$: $f(s+h) \to f(s)$ in L^1 for $h \to 0 \implies LHS \to 0$.

(2) lipschitz continuity: use the last estimate once more, where $f(t) \in C^{0,1}(I,X) \implies \|f(t)\|_X \leq C_1$, $\|f(t+h) - f(t)\| \leq C_2 |h|$, together: $\|v(t+h) - v(t)\| \leq C_0 C_1 h + C_0 T C_2 h = C_3 h$.

(3) differentiability:

$$\frac{1}{h}[v(t+h) - v(t)] = \underbrace{\frac{1}{h}\int_0^h S(t+h-s) f(s)\, ds}_{I_1(h)} + \underbrace{\int_0^t S(t-s) \left[\frac{f(s+h) - f(s)}{h}\right] ds}_{I_2(h)}$$

Let $t \in [0,T)$ be fixed, $h \to 0^+$: then $I_1(h) \to S(t) f(0) \ldots$ continuity of integrand (with respect to all variables, see ex. 6.4), $I_2(h) \to \int_0^t S(t-s) f'(s)\, ds \ldots$ since $\|S(t-s)\|_{\mathcal{L}(X)} \leq C_0$, we know: $\frac{f(s+h)-f(s)}{h} \to \frac{d}{dt} f(s) = f'(s)$ a. e. (see ex. 5.4, exchange of limit and integral: Lebesgue theorem, bounded majorant) $\ldots$ derivative from the right, derivative from the left is analogous $t \in (0,T]$. Together: $v'(t) = S(t) f(0) + \int_0^t S(t-s) f'(s)\, ds\ \forall t \in I$ and that is a continuous function of time t (property of c_0-semigroup + first part for f' instead of f), therefore $v(t) \in C^1(I,X)$.

Corollary: mild solution is neccessarily continuous.

Lemma 4.6 [Equivalent definition of mild solution]. A function $u(t)$ is a mild solution of (4.2) $\iff$ $u(t) \in C(I,X)$, $\int_0^t u(s)\, ds \in D(A)$, $u(t) = u_0 + A\left(\int_0^T u(s)\, ds\right) + \int_0^T f(s)\, ds$ for $\forall t \in I$.

Proof: **Step 1:**

"$\implies$" let $u(t) = S(t) u_0 + \int_0^t S(t-s) f(s)\, ds = u_1(t) + u_2(t)$ be a mild solution. (Remark: u_1 and u_2 are mild solutions of (4.2) for $f(t) = 0$ and $u_0 = 0$, respectively.)

ad u_1: $\int_0^t u_1(s)\, ds = \int_0^t S(s) u_0 ds \overset{\text{Th. 4.1, 3}}{\in} D(A)$ and $A\left(\int_0^t S(s) u_0 ds\right) = S(t) u_0 - u_0$.

ad u_2: consider $A\left(\int_0^t u_2(s)\,ds\right) = \dots$

$$\begin{aligned} \int_0^t u_2(s)\,ds &= \underbrace{\int_0^t \left(\int_0^s S(s-\sigma)\,f(\sigma)\,d\sigma\right) ds}_{\text{denote by } I_1} \\ \text{(Fubini: } 0<\sigma<s<t) &= \underbrace{\int_0^t \underbrace{\left(\int_\sigma^t S(s-\sigma)\,f(\sigma)\,ds\right)}_{\text{denote by } w(\sigma)} d\sigma}_{\text{designate } I_2} \end{aligned}$$

$w(\sigma) = \int_0^{t-\sigma} S(s)\underbrace{f(\sigma)}_{\text{fixed in } X} ds \in D(A)$. $Aw(\sigma) = \underbrace{S(t-\sigma)\,f(\sigma)}_{\in L^1(I,X)} - f(\sigma)\ \dots$ due to Th. 4.1, 3. Due to L. 4.4 we know: $A\left(\int_0^t u_2(s)\,ds\right) = \int_0^t Aw(\sigma)\,d\sigma = \int_0^t S(t-\sigma)\,f(\sigma) - f(\sigma)\,d\sigma = u_2(t) - \int_0^t f(s)\,ds$.
It is neccessary to check Fubini theorem for $(s,\sigma)\mapsto S(s-\sigma)\,f(\sigma)\in X$: we know that I_1, I_2 exist, it remains to show: $\langle f, I_1\rangle = \langle f, I_2\rangle\ \forall f\in X^*$, i. e., $\int_0^t\int_0^s \langle f, S(s-\sigma)\,f(\sigma)\rangle = \int_0^t\int_\sigma^t \langle f, S(s-\sigma)\,f(\sigma)\rangle \dots$ holds true due to (scalar) Fubini theorem.
together $\int_0^t u_1(s) + u_2(s)\,ds \in D(A)$,

$$A\left(\int_0^t u_1(s)+u_2(s)\,ds\right) = S(t)\,u_0 - u_0 + \underbrace{\int_0^t S(t-s)\,f(s)\,ds}_{u_2(t)} - \int_0^t f(s)\,ds$$

which was to be proven.

"$\Longleftarrow$" it suffices to show: the conditions in alternative definiton of mild solution are satisfied at most by one function $u(t)$ (since the mild solution satisfies these conditions due to previous implication). Let $u(t)$, $\widetilde{u}(t)$ satisfy the conditions. Denote $w(t) = u(t) - \widetilde{u}(t)$, therefore: $w(t)\in C(I,X)$, $\int_0^t w(s)\,ds \in D(A)$ and $w(t) = A\left(\int_0^t w(s)\,ds\right)\ \forall t\in I$. We will show that $w(t) = 0$ in I. Denote $U(t) = \int_0^t w(s)\,ds$, we know: $U(0) = 0$, $\frac{d}{dt}U(t) = w(t)$ (due to the continuity of $w(t)$), $U(t)\in D(A)$, $AU(t) = w(t)$. Therefore $U(t)$ is a classical solution of $\frac{d}{dt}u = Au$, $u(0) = 0$. Therefore it is zero, see the proof of L. 4.2 (see ex. 6.4, auxiliary function $y(t) = S(T-t)\,U(t)$, zero derivative).

Corollaries:

(1) Strong solution is a mild solution: proof: strong solution $u(t) = u_0 + \int_0^t Au(s)\,ds + \int_0^t f(s)\,ds$ for a. e. $t\in I$, $u(t)\in L^1(I, D(A))$, or more precisely its AC representative (due to L. 4.4 we can "pull" A outside the integral.
(2) There is at most one classical and/or strong solution. Proof: we know classical $\Longrightarrow$ strong $\Longrightarrow$ mild, which is unique. (Up to sets of zero measure for the strong solution.)

Remarks:

(1) The mild solution is not neccessarily a neither strong or classical solution. E. g., consider (4.2) for $u_0 = 0$, $f(t) = S(t)\,x$, where $x\in X$ is such that $S(t)\,x\notin D(A)$ for any $t>0$. I. e., the mild solution $u(t) = S(t)\,0 + \int_0^t \underbrace{S(t-s)\,S(s)}_{S(t)} x ds = tS(t)\,x \notin D(A)$, $t>0$ can be neither strong nor classical solution.
(2) Every mild solution is a (locally uniform) limit of very nice classical solutions.
Důkaz: let $u(t)$ be a mild solution, i. e., $u(t) = u_0 + A\left(\int_0^t u(s)\,ds\right) + \int_0^t f(s)\,ds$, $t\in I$. Define $u_h(t) = \frac{1}{h}\int_t^{t+h} u(s)\,ds$, $t\in[0,T-\Delta]$, $\Delta>0$, $h\in(0,\Delta)$. We know: $u_h(t) \rightrightarrows u(t)$ in $[0,T-\Delta]$ by the continuity of $u(t)$. Equation for $u_h(t)$? $\frac{d}{dt}u_h(t) = \frac{1}{h}[u(t+h) - u(t)] = \frac{1}{h}\left[A\left(\int_t^{t+h} u(s)\,ds\right) + \int_t^{t+h} f(s)\,ds\right] = Au_h(t) - f_h(t)$, where $f_h(t) = \frac{1}{h}\int_t^{t+h} f(s)\,ds$, $u_h(0)\in D(A)$.

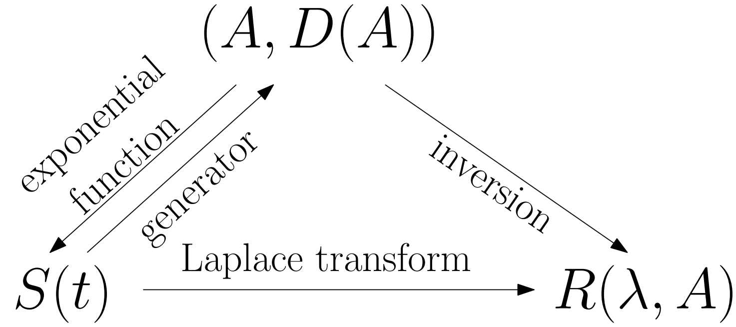

FIGURE 4.2.1. Relations between semigroup, generator and resolvent.

Theorem 4.4 [Regularity of mild solution].

(1) Let $u_0 \in D(A)$, $f(t) \in C^1(I, X)$. Then the mild solution is classical.
(2) Let $u_0 \in D(A)$, $f(t) \in C^{0,1}(I, X)$, moreover let X be reflexive. Then the mild solution is strong.

Proof: $u(t) = S(t)\,u_0 + \int_0^t S(t-s)\,f(s)\,ds = u_1(t) + v(t)$, $u_1(t)$ is classical solution (and therefore a strong one) for $f(t) \equiv 0$, it remains to show: $v(t)$ is a classical (strong) solution (4.2) for $u_0 = 0$. Let $f(t) \in C^{0,1}(I, X)$. Then due to L. 4.5, 2 we know: $v(t) \in C^{0,1}(I, X) \subset W^{1,1}(I, X)$, moreover $\exists \frac{d}{dt} v(t) \in X$ for a. e. $t \in I$ (see Chapter 1: Th. 1.5, use reflexivity).
Trick: $A(v(t)) = ?$

$$\begin{aligned}
\frac{1}{h}[S(h) - I]\,v(t) &= \frac{1}{h}\Big[\underbrace{S(h)\int_0^t S(t-s)\,f(s)\,ds}_{(*)} - \int_0^t S(t-s)\,f(s)\,ds\Big] \\
&= \frac{1}{h}[v(t+h) - v(t)] - \frac{1}{h}\int_t^{t+h} S(t+h-s)\,f(s)\,ds,
\end{aligned}$$

$$\begin{aligned}
\text{since } (*) &= \int_0^t S(t+h-s)\,f(s)\,ds \pm \int_t^{t+h} S(t+h-s)\,f(s)\,ds \\
&= v(t+h) - \int_t^{t+h} S(t+h-s)\,f(s)\,ds.
\end{aligned}$$

Take $h \to 0^+$: $RHS \to \frac{d}{dt} v(t) - f(t)$ ... due to the existence of derivative a. e. and the continuity of integrand. $(S(\underbrace{t+h-s}_{\to 0})f(s) = \varphi(h, s)$ ... continuous, $\varphi(0,0) = f(t))$.
I. e., for a. e. t: $v(t) \in D(A)$, $Av(t) = \frac{d}{dt} v(t) - f(t) \in L^1(I, X)$, L. 4.4 $\Longrightarrow$ $v(t) \in L^1(I, D(A))$, which was to be proven.
Part one is proven similarly, the derivative $Av(t) = \frac{d}{dt} v(t) - f(t)$ is evaluated everywhere, L. 4.5, 3, we will get a classical equation, $v(t)$ is continuous.

Remarks: Exponential formula $A \to S(t) =$ „e^{tA}".

(1) $A \in \mathcal{L}(X)$ ... $S(t) = e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \in \mathcal{L}(X)$ ... exactly all uniformly continuous semigroups, i. e., $S(t) \to I$ v $\mathcal{L}(X)$, $t \to 0^+$ (see ex. 5).
(2) Hille–Yosida (Th. 4.3): $S(t) = \lim_{n\to\infty} e^{-nt} e^{\overbrace{t\,n^2 R(n,A)}^{\in \mathcal{L}(X)}}$ ... "Yosida approximation".
(3) Statement: $S(t)$ is a c_0-semigroup, $\Longrightarrow S(t)\,u = \lim_{h\to 0^+} e^{tA_h u}$ ... holds for $\forall u \in X$ fixed, uniformly with respect to$t \in [0, T]$, where $A_h = \frac{1}{h}[S(h) - I]$.
An example of application: ex. 5.4: shift semigroup: $S(t) : u(x) \mapsto u(x+t)$, $u(x) \in L^2(\mathbb{R}) = X$, generator: $A = \frac{d}{dx}$ (weakly), $D(A) = W^{1,2}(\mathbb{R})$.
LHS: $[S(t)\,u](x) = u(x+t)$, *RHS*: $A_h = \frac{1}{h}[S(h) - I] : u(x) \mapsto \frac{u(h+h) - u(x)}{h} = d_h u(x)$.

Then $\left(e^{tA_h}u\right)(x) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\left(d_h\right)^k u\left(x\right)$, where $d_h^2 u\left(x\right) = d_h\left(d_h u\left(x\right)\right) = \frac{1}{h}\left(d_h\left(u+h\right) - d_h u\left(x\right)\right) = \frac{1}{h^2}\left(u\left(x+2h\right) + u\left(x\right) - 2u\left(x+h\right)\right)$.

Together: $u\left(x+t\right) = \lim_{h\to 0^+}\sum_{k=0}^{\infty}\frac{t^k}{k!}\left(d_h^k u\right)(x) \ldots$ Taylor (formally: if $d_h^k u\left(x\right) \to \left(\frac{d}{dx}\right)^k u\left(x\right)$, then we get an approximation by Taylor expansion, rigorously for $u\left(x\right)$ smooth.)

(4) Statement: $S(t)$ is a c_0-semigroup generated by $\left(A, D\left(A\right)\right)$, then $S(t)\,u = \lim_{n\to\infty}\left(I - \frac{t}{n}A\right)^{-n} u$ for $\forall u \in X$ fixed.

Remarks:
- an analogy of formula from basic analysis: $e^{at} = \lim_{n\to\infty}\left(1 - \frac{t}{n}a\right)^{-n}$
- expression on LHS makes sense: $\left(I - \frac{t}{n}A\right)^{-1} = \left(\frac{t}{n}\left(\frac{n}{t}I - A\right)\right)^{-1} = \frac{n}{t}R\left(\frac{n}{t}, A\right) \ldots$ well defined for n sufficiently large (see L. 4.3)

Def.: Let $S(t)$ be a c_0-semigroup on X generated by $\left(A, D\left(A\right)\right)$. We say that $S(t)$ is differentiable, if $S(t)\,u_0 \in D\left(A\right)$ for $\forall t > 0$ and $\forall u_0 \in X$.

Remark: It is easy to show that $S(t)\,u_0 \in D\left(A\right) \iff \lim_{h\to 0^+}\frac{1}{h}\left(S(t+h) - S(t)\right)u_0$ exists for$\forall t > 0$, $u_0 \in X$ and therefore $\lim \cdots = AS(t)\,u_0$. $\implies$ $S(t)\,u_0$ satisfies (4.1) classically for $\forall t > 0 \ldots$ "mollifying property".

Examples:
(1) Shift semigroup $S(t) : u\left(x\right) \mapsto u\left(x+t\right)$ is not differentiable since $S(t)\,u_0 \in D\left(A\right) = W^{1,2}\left(\mathbb{R}\right) \iff u_0 \in W^{1,2}\left(\mathbb{R}\right)$.
(2) Heat semigroup is differentiable (see below and ex. 6.3).

Remark: the heat semigroup is differentiable.
Proof: goal: $\mathcal{S}\left(t\right)u_0 = \sum_j \underbrace{e^{-\lambda_j t}u_j}_{\widetilde{u}_j} w_j \in D\left(\Delta\right) = W_0^{1,2} \cap W^{2,2}$ for $t > 0$, $u_0 \in L^2$ arbitrary. Due to statement 6.3.2, b) $\iff \sum_j \lambda_j^2 \widetilde{u}_j^2 < \infty \iff \sum_j \underbrace{\lambda_j^2 e^{-2\lambda_j t}}_{\le c_t} u_j^2 < \infty \ldots$ which follows by the followin:

- $\sum_j u_j^2 < \infty$
- $\lambda \mapsto \lambda^2 e^{-2\lambda t}$ is bounded with respect to $\lambda \in [0, \infty)$, $\lambda_j > 0$.

Remark:

$$(HE)\quad \begin{aligned} \frac{d}{dt}u &= \Delta u \\ u\left(0\right) &= u_0 \end{aligned}$$

solution from the point of view of chapter 4: define an operator $\Delta : D\left(\Delta\right) \to L^2$. We want to show that it is a generator ofc_0-semigroup (moreover, a semigroup of contractions) on L^2. Due to Th. 4.3 (Hille–Yosida) it suffices (and at the same time we know that it is neccessary) to check that:
- Δ is densely defined and closed.
- $\lambda I - \Delta : D\left(\Delta\right) \to L^2$ is injective and onto for $\forall \lambda > 0$
- $\big\| \underbrace{\left(\lambda I - \Delta\right)^{-1}}_{R(\lambda,\Delta)} f \big\| \le \frac{1}{\lambda}\left\|f\right\|_2$

(voluntary excercise), see below for the wave equation.

Example:

$$(WE)\quad \begin{aligned} \frac{d^2}{dt^2}u &= \Delta u \\ u\left(0\right) &= u_0 \\ \frac{d}{dt}u\left(0\right) &= u_1 \end{aligned}$$

we want to use the semigroup theory. How to deal with $\frac{d^2}{dt^2}$: d'Alembert: $U = \begin{pmatrix} u \\ v \end{pmatrix}$, $v = \frac{d}{dt}u \ldots$ (WE) is transformed to $\frac{d}{dt}U = AU$, $A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$. $X = W_0^{1,2} \times L^2$, $u_0 \in W_0^{1,2}$, $v \in L^2$, $D(A) = \underbrace{D(\Delta)}_{=W_0^{1,2} \cap W^{2,2}} \times W_0^{1,2}$.

Goal: A is a generator of c_0-semigroup of contractions on $X \ldots$ by Hille–Yosida.

- density: $D(A)$ is dense in $X \ldots$ $W_0^{1,2} \cap W^{2,2}$ is dense in $W_0^{1,2}$ and $W_0^{1,2}$ is dense in $L^2 \ldots$ known from previous courses on PDE.
- closedness: $U_n \in D(A)$, $AU_n = F_n$, $U_n \to U$, $F_n \to F$ in X, then $U \in D(A)$ and $AU = F$. component-wise: $U_n = \begin{pmatrix} u_n \\ v_n \end{pmatrix}$, $F_N = \begin{pmatrix} f_n \\ g_n \end{pmatrix}$, therefore $(WE) \iff \begin{matrix} v_n = f_n \ldots \text{ in } W_0^{1,2} \\ \Delta u_n = g_n \ldots \text{ in } L^2 \end{matrix}$. For $n \to \infty$: the first equation: $v = f \ \ldots$ in $W_0^{1,2}$, second equation: **trick:** a weak formulation $-(\nabla u_n, \nabla w) = (g_n, v)$, $n \to \infty$: $-(\nabla u, \nabla w) = (g, w)$. Use elliptic regularity: $-\Delta u = -g$ weakly, $g \in L^2$, elliptic regularity $\implies u \in W^{2,2}$.
- resolvent: we need: $\forall \lambda > 0, \forall F \in X\ \exists! U \in D(A) :\ (\lambda I - A) U = F$, $(U = R(\lambda, A) F)$, we want to show that $\|U\|_X \leq \frac{1}{\lambda} \|F\|_X$. Component-wise: $F = \begin{pmatrix} f \\ g \end{pmatrix} \in W_0^{1,2} \times L^2 \ldots \exists! U = \begin{pmatrix} u \\ v \end{pmatrix} \in W_0^{1,2} \cap W^{2,2} \times W_0^{1,2}$, i. e.:

$$\begin{aligned} \lambda u - v &= f \quad /\lambda\cdot, \text{ add to second equation} \\ \lambda v - \Delta u &= g \end{aligned}$$

Trick:

$$\iff \qquad \begin{aligned} v &= \lambda u - f \\ \lambda^2 u - \Delta u &= g + \lambda f \end{aligned}$$

There exists only one solution of second equation $u \in W_0^{1,2} \cap W^{2,2}$ (Lax-Milgram & elliptic regularity $\ldots$ weak formulation, $RHS \in L^2$) first equation: $v \in W_0^{1,2}$.

the "correct norm" on $X = W_0^{1,2} \times L^2$ is $\|(u, v)\| = \left(\|\nabla u\|_2^2 + \|v\|_2^2 \right)^{\frac{1}{2}}$. **Trick:**

$$\begin{aligned} \lambda u - v &= f \quad / - \Delta u, \int_\Omega \\ \lambda v - \Delta u &= g \quad /v, \int_\Omega \end{aligned}$$

$$\begin{aligned} \lambda \|\nabla u\|_2^2 - (\nabla v, \nabla u) &= (\nabla f, \nabla u) \\ \lambda \|v\|_2^2 + (\nabla u, \nabla v) &= (g, v) \end{aligned}$$

sum:

$$\begin{aligned} \lambda \left(\|\nabla u\|_2^2 + \|v\|_2^2 \right) &\leq \|\nabla f\|_2 \|\nabla u\|_2 + \|g\|_2 \|v\|_2 . \\ &\overset{\text{C.–S. in } \mathbb{R}^2}{\leq} \left(\|\nabla f\|_2^2 + \|g\|_2^2 \right)^{\frac{1}{2}} \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right)^{\frac{1}{2}} . \end{aligned}$$

Therefore $\lambda \left(\|\nabla u\|_2^2 + \|v\|_2^2 \right)^{\frac{1}{2}} \leq \left(\|\nabla f\|_2^2 + \|g\|_2^2 \right)^{\frac{1}{2}}$.

Example:

$$\begin{aligned} \frac{d}{dt}u &= Au + f(t) \\ u(0) &= u_0 \end{aligned} \tag{4.2}$$

$f(t) : I \to X$ and $u_0 \in X$ are given.

Recall: Mild solution: $u(t) = S(t)\,u_0 + \int_0^t S(t-s)\,f(s)\,ds$, $t \in I$, where $S(t)$ is a c_0-semigroup genereated by A.
Th. 4.4: $u_0 \in D(A)$, $f(t) \in C^{0,1}$ or C^1, then the mild solution is strong or classical, respectively.

Example: $\frac{\partial}{\partial t}u = \frac{\partial}{\partial x}u + f(t,x)$, $t > 0$, $x \in \mathbb{R}$, $u = u_0$ for $t = 0$, $x \in \mathbb{R}$. Transformation to (4.2) $\ldots$ $A = \frac{d}{dx}$, $D(A) = W^{1,2}(\mathbb{R})$, $u_0 \in X = L^2(\mathbb{R})$, $f(t,x) : I \leftrightarrow L^2(\mathbb{R})$, $t \mapsto f(t,\cdot)$. We know: $(A, D(A))$ is generator of shift c_0-semigroup: $[S(t)\,u] = u(x+t)$.
Mild solution: $u(t) = [S(t)\,u_0](x) + \int_0^t \underbrace{[S(t-s)\,f(s,\cdot)](x)}_{f(s,x+t-s)}\,ds = u_0(x+t) + \int_0^t f(s, x+t-s)\,ds \in L^2(\mathbb{R})$.

Remark: formally (i. e., classically for u_0, f smooth enough) it is a solution $\ldots$ (voluntary excercise), but the mild solution makes sense even for $u_0 \in L^2(\mathbb{R})$, $f(t) : I \to L^2$ integrable (and not better).

Excercises

Ex 1.1. Assume that $u(t) \in C(I,X)$ (always $I=[0,T]$) and prove the following propositions:

(1) $u(I) \subset X$ is compact, and $u(t): I \to X$ is uniformly continuous.
(2) Prove that $u(t): I \to X$ is strongly measurable (a) using the Pettis theorem and (b) directly from the definition
(3) Prove that $u(t): I \to X$ is Bochner integrable (a) using the Bochner theorem and (b) directly from the definition
(4) Show that $\int_I u(t)\,dt = \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^n u(jT/n)$

Ex 1.2.

(1) Let $u(t): I \to X$ be Bochner integrable, let $F: X \to Y$ be linear, continuous. Then $Fu(t): I \to Y$ is Bochner integrable, and
$$F\left(\int_I u(t)\,dt\right) = \int_I F\big(u(t)\big)\,dt$$
In particular: $\langle x^*, \int_I u(t)\,dt\rangle = \int_I \langle x^*, u(t)\rangle\,dt$ for any $x^* \in X^*$.
(2) Prove the following version of Fatou's lemma: let $u_n(t): I \to X$ are (strongly) measurable, and $u_n(t)$ converge weakly to $u(t)$ for a.e. $t \in I$. Then $u(t)$ is measurable and $\int_I \|u(t)\|\,dt \le \liminf_{n\to\infty} \int_I \|u_n(t)\|\,dt$. In particular, if $\int_I \|u_n(t)\|\,dt$ are bounded, then $u(t)$ is integrable.

Ex 1.3. Let $\psi_0(t): \mathbb{R} \to \mathbb{R}$ be convolution kernel, i.e. $\psi_0(t)$ is bounded, zero outside $[-1,1]$ and $\int_{-1}^1 \psi_0(t)\,dt = 1$. Let $u(t): I \to X$ be given, and assume that $u(t)$ is extended by zero outside of I. Let $\psi_n(t) = n\psi_0(nt)$ and finally let $u_n(t) = u * \psi_n(t) = \int_{\mathbb{R}} u(t-s)\psi_n(s)\,ds$.

(1) Show that $u_n(t) \in C(I,X)$ and if $\psi_0 \in C^1$, then also $u_n(t) \in C^1(I,X)$ and $u_n'(t) = u * \psi_n'(t)$.
(2) Show that norm of $u_n(t)$ is not larger than the norm of $u(t)$ in the spaces $C(I,X)$, $L^p(I,X)$.
(3) Show that if $u(t) \in C(J,X)$ for some J strictly larger than I, then $u_n(t) \rightrightarrows u(t)$ in $C(I,X)$, for $n \to \infty$.

Ex 1.4.

(1) Prove the following Convergence Principle: Let $F_n: X \to X$ be a sequence of linear operators such that the norms $\|F_n\|$ are bounded independently of n. Let there be a dense $S \subset X$ such that $F_n v \to v$ as $n \to \infty$ for any $v \in S$. Then $F_n u \to u$ as $n \to \infty$ for any $u \in X$.
(2) Apply the Convergence Principle to prove part 4 of Lemma 1.1.

Ex 2.1. Let X be reflexive, separable.

(1) Let $p \in (1,\infty)$. Show that any $u(t) \in W^{1,p}(I;X)$ has a α-Hölder continuous representative, with $\alpha = 1 - 1/p$.
(2) Show that $W^{1,\infty}(I;X) = C^{0,1}(I;X)$ (the space of Lipschitz functions), in the sense of representative.
(3) Let $u_n(t)$ be weakly differentiable, and let $u_n(t) \rightharpoonup u(t)$, $\frac{d}{dt}u_n(t) \rightharpoonup g(t)$ (weakly) in $L^1(I;X)$. Then $u(t)$ is weakly differentiable, with $\frac{d}{dt}u(t) = g(t)$.
(4) Let $u_n(t)$ are bounded in $L^p(I;Y)$, $\frac{d}{dt}u_n(t)$ are bounded in $L^q(I;Z)$, where $p,\,q \in (1,\infty)$ and Y, Z are reflexive, separable. Then there is a subsequence so that $\tilde{u}_n(t \rightharpoonup u(t)$, $\frac{d}{dt}\tilde{u}_n(t) \rightharpoonup g(t)$ in the respective spaces, and $\frac{d}{dt}u(t) = g(t)$.

Ex 2.2.

(1) Prove that $L^p(I;L^p(\Omega)) = L^p(I\times\Omega)$ if $p \in [1,\infty)$, but $L^\infty(I;L^\infty(\Omega)) \subsetneq L^\infty(I\times\Omega)$.
(2) Prove that if $u_n \rightharpoonup u$ in $L^p(I;L^q(\Omega))$, and $\Omega \subset \mathbb{R}^n$ is open, bounded, then
$$\int_{I\times\Omega} u_n(t,x)\psi(t,x)\,dtdx \to \int_{I\times\Omega} u(t,x)\psi(t,x)\,dtdx$$
for any (say) bounded, measurable function $\psi(t,x)$.

Ex 2.3. Let H be a Hilbert space. Show that H is uniformly convex. Show directly that if $u_n \rightharpoonup u$ and $\|u_n\|_H \to \|u\|_H$, then $u_n \to u$.

*** Ex 2.4.** Let $u_n(t)$ be bounded in $L^p(I;X)$, where $p \in (1,\infty]$, and X be reflexive, separable. Prove that there is a weakly convergent ($*$-weak if $p=\infty$) subsequence, using only Theorem 1.9 and separability of $L^{p'}(I;X^*)$.

Ex 2.5. Let $u_n(t) \rightharpoonup u(t)$ in $L^p(I;X)$, $v_n(t) \to v(t)$ in $L^{p'}(I;X^*)$, where p, p' are Hölder conjugate. Prove that $\int_I \langle u_n(t) v_n(t)\rangle_{X,X^*}\,dt \to \int_I \langle u(t), v(t)\rangle_{X,X^*}\,dt$.

Ex 3.1. Let $u(t) \in L^2(I;W_0^{1,2})$, $g(t) \in L^2(I;W^{-1,2})$ and $u_0 \in L^2$. Then the following are equivalent:

(i) $\frac{d}{dt}u(t) = g(t)$ and $u(0) = u_0$ (in the sense of representative)
(ii) for any $v \in W_0^{1,2}$, $\varphi \in C_c^\infty((-\infty,T))$ one has

$$-\int_I (u(t),v)\varphi'(t)\,dt = \int_I \langle g(t),v\rangle\,\varphi(t)\,dt + (u_0,v)\varphi(0)$$

Corollary: $u\left(t\right) \in L^2\left(I, W_0^{1,2}\right)$ is w. s. with $u\left(0\right) = u_0$ (for representative) iff

$$\int_I \left(u\left(t\right),v\right)\varphi'\left(t\right)dt + \int_I \left(a\left(\nabla u\left(t\right)\right),v\right)\varphi\left(t\right)dt + \int_I \left(f\left(u\left(t\right)\right),v\right)\varphi dt$$
$$= \int_I \left\langle h\left(t\right),v\right\rangle + \left(u_0,v\right)\varphi\left(0\right)$$

Ex 3.2. Recall the notation and assumptions from Chapter 2: let $f(z):\mathbb{R}\to\mathbb{R}$, $a(\xi):\mathbb{R}^n\to\mathbb{R}^n$ be Lipschitz continuous. Let the operators $\mathcal{A}: W_0^{1,2} \to W^{-1,2}$ and $\mathcal{F}: I\times W_0^{1,2} \to W^{-1,2}$ be defined as

$$\langle \mathcal{A}(u), v\rangle = \int_\Omega a(\nabla u(x))\cdot\nabla v(x)\,dx$$
$$\mathcal{F}(t,u) = -\mathcal{A}(u) - \iota f(u) + h(t)$$

where $h(t) \in L^2(I;W^{-1,2})$ is a fixed function.

(1) Prove that $u \mapsto f(u)$ is Lipschitz as operator $L^2 \to L^2$, and also $u(t) \mapsto f(u(t))$ is Lipschitz as operator $L^2(I;L^2) \to L^2(I;L^2))$. N. B. do not forget to verify that $f(u)$ and $f(u(t))$ are *measurable* in the appropriate sense.
(2) Deduce that also $u(t) \mapsto \iota f(u(t))$ is Lipschitz as operator $L^2(I;W_0^{1,2}) \to L^2(I;W^{-1,2})$.
(3) Prove that $u \to \mathcal{A}(u)$ is Lipschitz continuous as operator $W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$.
(4) Show that $\|\mathcal{F}(t,u)\|_{-1,2} \le c(1+\|u\|_{1,2}+\|h(t)\|_{-1,2})$ with some constant only depending on the nonlinearities $a(\cdot)$ and $f(\cdot)$.

Ex 3.3. Let $W_0^{1,2} \hookrightarrow L^2 \hookrightarrow W^{-1,2}$ be the Gelfand triple, with the embedding $\iota: W_0^{1,2} \to W^{-1,2}$.

(1) Observe that due to the Poincaré inequality, $W_0^{1,2}$ is a Hilbert space with the scalar product $((u,v)) = \int_\Omega \nabla u\left(x\right)\cdot\nabla v\left(x\right)\,dx$.
(2) By Riesz theorem, any $f \in W^{-1,2}$ can be represented by some $u_f \in W_0^{1,2}$ so that

$$\langle f,v\rangle = ((u_f,v)) \quad \forall v \in W_0^{1,2}$$

(3) Show that by a Green formula $((u,v)) = (-\Delta u, v)$ for any $v \in W_0^{1,2}$ and $u \in C_c^\infty$.
(4) Combine that above with the density of C_c^∞ in $W_0^{1,2}$ to show that for any $f \in W^{-1,2}$ there exists smooth functions u_n such that $\iota u_n \to f$.
(5) Finally, show that $\iota: W_0^{1,2} \to W^{-1,2}$ is *injective*.

These are the reasons why no symbol "ι" is normally employed and $(\cdot,\cdot)_{L^2}$ is simply seen as a generalization of $\langle\cdot,\cdot\rangle_{W^{-1,2},W_0^{1,2}}$ without further notational ado.

Ex 4.1. Let X be reflexive, separable, $X \hookrightarrow Z$, and let $u(t) \in L^\infty(I;X)\cap C(I;Z)$. Show that $u(t) \in X$ for *all* $t \in I$ and moreover, $t \mapsto u(t) \in X$ is weakly continuous.

Ex 4.2. Let w_j be the eigenfunctions of $-\Delta u = \lambda u$ with zero Dirichlet b.c. Let P_N be the ON projection (in L^2) on the space span $\{w_1, \dots, w_N\}$. Clearly P_N is continuous $L^2 \to L^2$ with norm 1.

(1) Show that P_N is also continuous $W_0^{1,2} \to W_0^{1,2}$ with norm 1, if $W_0^{1,2}$ is taken as a Hilbert space with scalar product $(\!(u,v)\!) = (\nabla u, \nabla v)$.
(2) Show that $\|P_N u\|_{2,2} \le c \|u\|_{2,2}$ for any $u \in W_0^{1,2} \cap W^{2,2}$ (assume $\partial\Omega$ sufficiently regular).

Ex 4.3. Let $\psi(z) : \mathbb{R} \to \mathbb{R}$ be smooth function with a bounded derivative. Show that $u_n \to u$ in $W^{1,2}$ implies $\psi(u_n) \to \psi(u)$ in $W^{1,2}$.

Ex 4.4. [d'Alembert's transform]. Let $u(t) : I \to X$, $g(t) : I \to X$ be integrable functions. Then the following assertions are equivalent:
(i) $\frac{d^2}{dt^2} u(t) = g(t)$ weakly, i.e.

$$\int_I u(t)\varphi''(t)\,dt = \int_I g(t)\varphi(t)\,dt \qquad \forall \varphi(t) \in C_c^\infty(I)$$

(ii) there is $v(t) : I \to X$ integrable such that $\frac{d}{dt} u(t) = v(t)$ and $\frac{d}{dt} v(t) = g(t)$ weakly in I.

Ex 5.1. Let $S(t)$ be a c_0-semigroup in X. Show that the following are equivalent:

(1) $S(t) = e^{tA}$ for some $A \in \mathcal{L}(X)$
(2) $S(t)$ is uniformly continuous, i.e. $S(t) \to I$ in $\mathcal{L}(X)$ for $t \to 0+$

Ex 5.2. Let $u(t) \in L^2(I; W_0^{1,2}) \cap C(I; L^2)$ be the (unique) weak solution to the heat equation

$$\frac{d}{dt} u - \Delta u = 0, \qquad u(0) = u_0$$

Verify that the solution operators $S(t) : u_0 \mapsto u(t)$ form a c_0-semigroup in L^2.

Ex 5.3. Let $(A, \mathcal{D}(A))$ be an unbounded operator in X, which is closed, and let $\mathcal{D}(A)$ be dense in X.

(1) Let $v'(t) = \lim_{h\to 0} \frac{1}{h}(v(t+h) - v(t)$ be the classical derivative in X. Assuming that $u'(t)$ and $(Au)'(t)$ exist, show that $u'(t) \in \mathcal{D}(A)$ and $A(u'(t)) = (Au)'(t)$.
(2) Assume that $u(t) : I \to \mathcal{D}(A)$ be Bochner integrable, where $\mathcal{D}(A)$ is equipped with the graph-norm $\|u\|_X + \|Au\|_X$.
Show that both $u(t) : I \to X$ and $Au(t) : I \to X$ are Bochner integrable, and $A\big(\int_I u(t)\,dt\big) = \int_I Au(t)\,dt$.

Ex 5.4. Let $X = L^2(\mathbb{R})$ and define the "shift" operators $S(t) : X \to X$ by $S(t) : f(x) \mapsto f(x+t)$.

(1) Verify that $S(t)$ form a c_0-semigroup
(2) Show that $\|S(t) - I\|_{\mathcal{L}(X)} = 2$ for any $t > 0$, hence the semigroup is not uniformly continuous
(3) Prove that if $f(x) \in W^{1,2}(\mathbb{R})$, then $\frac{1}{h}(S(h)f(x) - f(x)) \to \frac{d}{dx} f(x)$ in $L^2(\mathbb{R})$, as $h \to 0^+$.
(4) Prove conversely that if $f(x)$, $g(x) \in L^2(\mathbb{R})$ are such that $\frac{1}{h}(S(h)f(x) - f(x)) \to g(x)$ in $L^2(\mathbb{R})$, as $h \to 0^+$, then $f(x) \in W^{1,2}(\mathbb{R})$ and $\frac{d}{dx} f(x) = g(x)$
(5) Observe that the above assertions imply that the generator of $S(t)$ is the operator $A : f(x) \mapsto \frac{d}{dx} f(x)$ with the domain of definition $\mathcal{D}(A) = W^{1,2}(\mathbb{R})$.

Ex 6.1. Consider space ℓ^2 of sequences $\{u_j\}$ with the norm $\big(\sum_j u_j^2\big)^{1/2}$.
Let λ_j be real numbers. Define operator $A : D(A) \subset \ell^2 \to \ell^2$ as

$$A : \{u_j\} \mapsto \{\lambda_j u_j\}, \qquad D(A) = \left\{ \{u_j\} \in \ell^2;\ \sum_j \lambda_j^2 u_j^2 < \infty \right\}$$

(1) Observe that $D(A) = \ell^2$ and $A \in \mathcal{L}(\ell^2) \iff$ the sequence $\{\lambda_j\}$ is bounded.
(2) Show that $(A, D(A))$ is closed, and $D(A)$ is dense in ℓ^2.
(3) Assume that $\lambda_j \le \omega$ for all j. Deduce that any $\lambda > \omega$ belongs to the resolvent set $\rho(A)$. Write an explicit formula for the resolvent $R(\lambda, A)$.

(4) By Hille-Yosida theorem, A is a generator of a c_0-semigroup in ℓ^2, which satisfies $\|S(t)\|_{\mathcal{L}(\ell^2)} \le e^{\omega t}$, $t \ge 0$. Compute $S(t)$ explicitely.

Ex 6.2. Let $S(t)$ be a c_0-semigroup in X, satisfying $\|S(t)\|_{L(X)} \le Me^{\omega t}$, $t \ge 0$.
Let $\tilde{S}(t) = e^{-\omega t}S(t)$. Prove that:

(1) $\tilde{S}(t)$ is also a c_0-semigroup, and $\left\|\tilde{S}(t)\right\|_{L(X)} \le M$, $t \ge 0$.
(2) Let $(A, D(A))$, $\tilde{A}, \mathcal{D}(\tilde{A}))$ be the generators of $S(t)$, $\tilde{S}(t)$, respectively. Show that $D(A) = \mathcal{D}(\tilde{A})$ and $\tilde{A} = A - \omega I$.

***Ex. 6.3.** Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary. Let λ_j, w_j be the eigenvalues and eigenfunctions of the Dirichlet laplacian, with $\|w_j\|_2 = 1$.
By Parseval's theorem, we have $u_0 = \sum_j u_j w_j$, where $u_j = (u, w_j)$ and the sum converges in L^2.

(1) Show that $u(t) = \sum_j e^{-\lambda_j t} u_j w_j$ is a weak solution to the heat equation $\frac{d}{dt}u - \Delta u = 0$, $u(0) = u_0$.
(2) Show that $\sum_j \lambda_j u_j^2 < \infty$ if and only if $u_0 \in W_0^{1,2}$. Show that $\sum_j \lambda_j^2 u_j^2 < \infty$ if and only if $u_0 \in W_0^{1,2} \cap W^{2,2}$.
(3) Observe that the "heat semigroup" (see Exercise 5.2) can thus be identified with a multiplicative semigroup (as in Exercise 6.1). It follows that the generator of heat semigroup is the operator $\Delta = \sum_{i=1}^d \frac{\partial^2 u}{\partial x_i^2}$, with derivatives understood weakly on the domain of definition $\mathcal{D}(\Delta) = W_0^{1,2} \cap W^{2,2}$.

Ex 6.4. Let $S(t)$ be a c_0-semigroup on X.

(1) Show that the map $(t, x) \mapsto S(t)x$ is *jointly* continuous $[0, \infty) \times X \to X$.
(2) Assuming that $\tilde{S}(t)$ is another c_0-semigroup, show that $y(t) = S(T - t)\tilde{S}(t)x$ is continuous $[0, T] \to X$, where $T > 0$ and $x \in X$ are fixed.
(3) Assuming finally that $S(t)$ and $\tilde{S}(t)$ have the same generator, show that $y'(t) = 0$ for any $t \in (0, T)$ if $x \in D(A)$.
(4) Deduce that $S(t) = \tilde{S}(t)$ for all $t \ge 0$, thus establishing Lemma 4.2.

Hints

Ex 1.1.

(1) Follows from compactness of I just as in the scalar case $X = \mathbb{R}$.
(2) (a) compact implies separable, and continuous scalar is measurable; (b) set $u_n(t) = u(jT/n)$ for $t \in [(j-1)T/n, jT/n]$ – these are simple functions and $u_n(t) \rightrightarrows u(t)$ thanks to uniform continuity
parts 3 and 4 use very similar ideas

Ex 1.2. 1. Let $u_n(t)$ be simple functions from the definition of $\int_I u(t)\,dt$. Then $Fu_n(t)$ are simple… In particular: set $Y = \mathbb{R}$.
2. Use the fact that a separable set can be enlarged to a closed and convex (hence weakly closed) set. Use weak lower semicontinuity of the norm and scalar version of Fatou's lemma.

Ex 1.3. 1. Rewrite $u_n(t) = \int_{\mathbb{R}} u(s)\psi_n(t-s)\,ds$ and show that usual theorems about dependence of integral on parameter apply. (In fact for ψ_0 smooth enough, the dependence on t is uniform, so the exchange of integral and limit is trivial.)
2. For $p = 1$, this follows by Fubini's theorem.
3. Use uniform continuity of $u(t)$ on the neighborhood of I.

Ex 1.4. 1. Fix $u \in X$, and let $\varepsilon > 0$ arbitrary be given. Pick $v \in S$ such that $\|u - v\| < \varepsilon$. Write $F_n u - u = F_n(u - v) + \big(F_n v - v\big) + (v - u)$ and show that each term is estimated by (a multiple of) ε if n is large enough.
2. Set $X = L^p(I; X)$, $F_n u = u * \psi_n$ and $S = C_c(I; X)$. Use the results of Ex 1.3.2 and 1.3.3.

Ex 2.1.

(1) By Lemma 1.5, there is a continuous representative $\tilde{u}(t)$ such that $\tilde{u}(t_1)-\tilde{u}(t_2) = \int_{t_1}^{t_2} \frac{d}{dt}u(s)\,ds$. Estimate the integral using the Hölder inequality.
(2) Inclusion $\subset$ is as above. For the converse, note that Lipschitz function is absolutely continuous, and its derivative is L^∞, cf. Theorem 1.5.
(3) Explain (in detail), that weak convergence is enough to pass in the definition of the weak derivative.
(4) Use Eberlein-Šmulian and the previous problem.

Ex 2.3.

(1) Use $\left\|\frac{x+y}{2}\right\|_H^2 + \left\|\frac{x-y}{2}\right\|_H^2 = \|x\|_H^2 + \|y\|_H^2$.
(2) Write $\|u-u_n\|_H^2 = \|u\|_H^2 - 2(u,u_n)_H + \|u_n\|_H^2$.

Ex 2.4. Let $v_n(t)$ be a countable dense set in $L^{p'}(I;X^*)$ ...

Ex 2.5. Add and subtract $\langle u_n(t), v(t)\rangle$.

Ex 4.1. $\exists K>0$, $N\subset I$ s.t. $\lambda(N)=0$ and $\|u(t)\|_X \le K$ for all $t\in I\setminus N$. Approximate $t_0\in N$ with $t_n\to t_0$, $t_n\in I\setminus N$ to show that $\|u(t_0)\|_X\le K$. Prove continuity by contradiction, using uniqueness of limits in Z.

Ex 4.2.

(i) Rewrite P_N as ON (in $W_0^{1,2}$ w.r. to $(\!(\cdot,\cdot)\!)$) projection
(ii) Show that $P_N(-\Delta u) = -\Delta P_N u$; use elliptic regularity for the laplacian

Ex 4.3. In view of Lemma 2.4, it is enough to show that $u_n\to u$, $\nabla u_n\to\nabla u$ in L^2 implies $\psi'(u_n)\nabla u_n\to\psi'(u)\nabla u$ in L^2. By taking a subsequence we can in the first step assume $u_n\to u$ a.e. Show further by contradiction (and step one) that convergence takes place even without taking a subsequence.

Ex. 5.3.2. Let $u_n(t)$ be simple functions and $u_n(t)\to u(t)$ in the norm of $\mathcal{D}(A)$ for a.e. $t\in I$, ...

Ex. 5.4.

2. Consider suitable $f(x)\in L^2(\mathbb{R})$ with compact support.
3. Working with AC representative, we have $f(x+h)-f(x) = \int_0^h g(x+s)\,ds$, where $g=\frac{d}{dx}f$. Deduce that $\frac{1}{h}\big(f(x+h)-f(x)\big)$ can be written as convolution of g with suitable kernels, and use Lemma 1.1, part 4.
4. Let $\varphi(x)\in C_c^\infty(\mathbb{R})$ be given test function and $h>0$ be fixed. Prove that

$$\int_{\mathbb{R}} \frac{f(x+h)-f(x)}{h}\varphi(x)\,dx = \int_{\mathbb{R}} f(x)\frac{\varphi(x-h)-\varphi(x)}{h}\,dx$$

   Using the assumptions, show that you can take the limit $h\to 0+$ on both sides, to obtain that $\frac{d}{dx}f(x) = g(x)$ in the sense of weak derivative.

Ex. 6.1. in part 2, for density, consider $\mathcal{C}\subset D(A)$ consisting of $\{u_j\}$ with only finitely many nonzero u_j, for 3. $R(\lambda,A): \{v_j\}\mapsto\{\frac{1}{\lambda-\lambda_j}v_j\}$, in 4. use Yosida approximation.
Alternatively, one can guess that the answer $S(t):\{u_j\}\mapsto\{e^{\lambda_j t}u_j\}$ and then verify that $S(t)$ is a c_0-semigroup on ℓ^2 and A is its generator. So $S(t)$ must be the sought-for semigroup by Lemma 4.2.
For taking limits in ℓ^2, you can use the following version of Lebesgue's theorem (where sum is seen as an integral): if $\sum_j |a_j|<\infty$ and $b_j(t)$ are bounded independently of j and t, and $b_j(t)\to\beta_j$, then $\sum_j b_j(t)a_j\to\sum_j\beta_j a_j$.

Ex. 6.4. for 1, use part 1 of Lemma 4.1; in writing $y(t+h)-y(t)$, add $\pm S(T-(t+h))\tilde{S}(t)x$ and use joint continuity and definition of the generator, and Theorem 4.1; in 4. $S(T)x=\tilde{S}(T)x$ if $x\in D(A)$ and use density.

Solutions

Ex. 1.2.

(1) Let us first prove that $Fu(t)$ is Bochner integrable. Since $u(t)$ is Bochner integrable, there exist simple functions $u_n(t) = \sum_{j=1}^{k(n)} \chi_{A_j^n}(t)x_j^n$ such that $u_n(t) \to u(t)$ for a. e. $t \in I$ and $\int_I \|u(t) - u_n(t)\|_x \, dt \to 0$. Denote $s_n(t) = \sum_{j=1}^{k(n)} \chi_{A_j^n}(t)F(x_j^n)$. Then s_n are clearly simple. By linearity of F we have $s_n(t) = F(u_n(t))$. Since F is also continuous, it holds that

$$\lim_{n\to\infty} s_n(t) = F\left(\lim_{n\to\infty} u_n(t)\right) = Fu(t)$$

for a. e. $t \in I$. By continuity and linearity of F we have ($\|F\| < \infty$)

$$\int_I \|Fu(t) - Fu_n(t)\|_Y \, dt \le \|F\| \int_I \|u(t) - u_n(t)\|_X \, dt \to 0.$$

Thus we see that $Fu(t)$ is Bochner integrable.

It follows from the definition of Bochner integral and continuity and linearity F that

$$\begin{aligned} F\left(\int_I u(t)\,\mathrm{dt}\right) &= F(\lim_{n\to\infty} \int_I u_n(t)\,dt) = \lim_{n\to\infty} F\left(\int_I u_n(t)\,dt\right) \\ &= \lim_{n\to\infty} F\left(\sum_{j=1}^{k(n)} \lambda(A_j^n)x_j^n\right) = \lim_{n\to\infty} \sum_{j=1}^{k(n)} \lambda(A_j^n)F(x_j^n) \\ &= \lim_{n\to\infty} \int_I s_n(t)\,dt = \int_I Fu(t)\,dt. \end{aligned}$$

In particular, if we choose $Y = \mathbb{R}$, we obtain for every $x^* \in X^*$

$$\langle x^*, \int_I u(t)\,dt\rangle = \int_I \langle x^*, u(t)\rangle\,dt.$$

(2) We begin by proving the measurability of $u(t)$. Let $x^* \in X^*$ be arbitrary. Since $u_n(t)$ are measurable and therefore weakly measurable, the scalar functions $\langle x^*, u_n(t)\rangle$ are measurable. But since $u_n(t) \rightharpoonup u(t)$ for a. e. $t \in I$, we have

$$\langle x^*, u_n(t)\rangle \to \langle x^*, u(t)\rangle$$

for a. e. $t \in I$. Scalar function $\langle x^*, u(t)\rangle$ is for a. e. $t \in I$ pointwise limit of sequence of scalar measurable functions and therefore it is a scalar measurable function. As $x^* \in X^*$ was chosen arbitrarily, the function $u(t)$ is weakly measurable.

Since $u_n(t)$ are measurable and therefore λ-separably valued, there exist Lebesgue null sets N_n such that $u_n(I \setminus N_n)$ are separable sets. Moreover, there exists Lebesgue null null set N_0 such that $u_n(t) \rightharpoonup u(t)$ for a. e. $t \in I \setminus N_0$. Denote

$$N = \bigcup_{k=0}^{\infty} N_k,$$

which is also a Lebesgue null set. Therefore $u_n(I \setminus N)$ are separable sets and for every $t \in I \setminus N$ we have that $u_n(t) \rightharpoonup u(t)$. As a countable union of separable sets is a separable set, the union

$$\bigcup_{n=1}^{\infty} u_n(I \setminus N)$$

is a separable set and thus span $\left(\bigcup_{n=1}^{\infty} u_n(I \setminus N)\right)$ is a separable subspace. Since subspace is clearly a convex subset, it follows easily that

$$u(t) \in \overline{\bigcup_{n=1}^{\infty} u_n(I \setminus N)}^{w} \subseteq \overline{\mathrm{span}(\bigcup_{n=1}^{\infty} u_n(I \setminus N))}^{w} = \overline{\mathrm{span}(\bigcup_{n=1}^{\infty} u_n(I \setminus N))}^{\|\cdot\|_X}$$

for every $t \in I \setminus N$. Therefore $u(t)$ is also λ-separably valued and measurable by the Pettis theorem.

By weak lower semicontinuity of the norm and by the scalar version of Fatou's lemma it holds that

$$\int_I \|u(t)\| \, dt \le \int_I \liminf_{n\to\infty} \|u_n(t)\| \, dt \le \liminf_{n\to\infty} \int_I \|u_n(t)\| \, dt.$$

Moreover, if the sequence $\int_I \|u_n(t)\| \, dt$ is bounded, it follows easily that $\int_I \|u(t)\| \, dt \le \liminf_{n\to\infty} \int_I \|u_n(t)\| \, dt < \infty$, therefore $u(t)$ is integrable by the Bochner theorem.

Ex. 1.3. Convolution kernel: $\psi_0(t) : \mathbb{R} \to \mathbb{R}$, bounded, measurable, nonnegative, $\psi_0(t) = 0$ outside $[-1,1]$, $\int_{\mathbb{R}} \psi_0(t)\,dt = 1$. Define $\psi_n(t) = n\psi_0(nt)$, $u_n(t) = u * \psi_n(t) = \int_{\mathbb{R}} u(t-s)\,\psi_n(s)\,ds = \int_{\mathbb{R}} \psi_n(t-s)\,u(s)\,ds$, we set $u(t) = 0$ outside of I.

(1) $u(t) \in L^1(I;X)$, $\psi_0(t) \in C^1 \implies u_n(t) \in C^1(I,X)$ and $u_n'(t) = u * \psi_n'(t)$ for any n fixed.
Is $u_n(t)$ continuous? Let $t \in I$ be fixed. $h_j \to 0 \implies u_n(t+h_j) \to u_n(t)$, but $u_n(t+h_j) - u_n(t) = \int_I \underbrace{[\psi_n(t+h_j-s) - \psi_n(t-s)]\,u(s)}_{v_j(s)}\,ds$. By Th. 1.3 (Lebesgue) we have: $v_j(s) \to 0$, $j \to \infty$ for a. e. $s \in \mathbb{R}$ fixed and $\|v_j(s)\| \le (|\psi_n(\ldots)| + |\psi_n(\ldots)|)\,\|u(s)\| \le c\,\|u(s)\| \in L^1(I;\mathbb{R})$.
Is u differentiable?

$$\begin{aligned} u_n'(t) &= \frac{d}{dt}\int_I \psi_n(t-s)\,u(s)\,ds \\ &= \lim_{h\to 0}\int_I \underbrace{\left[\frac{\psi_n(t+h-s) - \psi_n(t-s)}{h}\right]}_{\to \psi_n'(t-s) \text{ for } \forall s \text{ fixed}} u(s)\,ds \end{aligned}$$

$$|[\ldots]| \le C \ldots \text{ Lagrange mean value theorem } (C = \sup|\psi_n'|)$$

$$= \int_I \psi_n'(t-s)\,u(s)\,ds$$

(2) $\|u_n(t)\|_{\mathcal{F}} \le \|u(t)\|_{\mathcal{F}}$ $\forall n$, where $\mathcal{F} = C(I,X)$ or $L^p(I;X)$.
For L^p:

- $p = 1$:

$$\begin{aligned} \|u_n(t)\|_{L^1(I;X)} &= \int_I \|u * \psi_n(t)\|_X\,dt = \int_I \left\| \int_{\mathbb{R}} u(s)\,\psi_n(t-s)\,ds \right\|_X dt \\ &\le \int_I \left(\int_{\mathbb{R}} \|u(s)\|_X\,\psi_n(t-s)\,ds \right) dt \\ &\overset{\text{Fubini}}{=} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \|u(s)\|_X\,\psi_n(t-s)\,dt \right) ds \\ &= \int_{\mathbb{R}} \|u(s)\| \underbrace{\left(\int_{\mathbb{R}} \psi_n(t-s)\,dt \right)}_{=1} ds \\ &= \int_{\mathbb{R}} \|u(s)\|\,ds = \|u(\cdot)\|_{L^1(I;X)} \end{aligned}$$

- for $p = \infty$ easy (same as $C(I,X)$).
- $p \in (1,\infty)$: Hölder inequality: $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$

$$\int_\Omega |fg| \le \left(\int_\Omega |f|^p \right)^{\frac{1}{p}} \left(\int_\Omega |g|^{p'} \right)^{\frac{1}{p'}}.$$

Trick:

$$\begin{aligned}\|u*\psi_n(t)\| &\le \int_{\mathbb{R}} \|u(s)\|\,\psi_n(t-s)\,ds \\ &= \int \underbrace{\|u(s)\|\,(\psi_n(t-s))^{\frac{1}{p}}}_{f(s)}\underbrace{(\psi_n(t-s))^{1-\frac{1}{p}}}_{g(s)}\,ds \\ &\overset{\text{Hölder}}{\le} \left(\int_{\mathbb{R}} \|u(s)\|^p\,\psi_n(t-s)\,ds\right)^{\frac{1}{p}} \underbrace{\left(\int_{\mathbb{R}} (\psi_n(t-s))\,ds\right)^{\frac{1}{p'}}}_{=1}\end{aligned}$$

$\Longrightarrow$ $\|u*\psi_n(t)\|^p \le \int_{\mathbb{R}} \|u(s)\|^p\,\psi_n(t-s)\,ds$, now as before: $\int_{\mathbb{R}} dt$, Fubini

$$\Longrightarrow \int_{\mathbb{R}} \|u*\psi_n(t)\|^p\,dt \le \int_{\mathbb{R}} \|u(s)\|^p\,ds.$$

(3) If $u(t)\in C(J,X)$, J compact larger than I, then $u_n(t)\rightrightarrows u(t)$ in I.

$$\begin{aligned}\|u_n(t)-u(t)\| &= \left\|\int_{\mathbb{R}} u(s)\,\psi_n(t-s)\,ds-u(t)\right\| \\ &\le \int_{\mathbb{R}} \|u(s)-u(t)\|\,\underbrace{\psi_n(t-s)}_{\neq \text{ only if } |t-s|\le\frac{1}{n}}\,ds\end{aligned}$$

($t\in I$, we only consider $s\in\left[-\frac{1}{n},T+\frac{1}{n}\right]\subset J$ for n large). Uniform continuity of $u(t)$ on J: $\varepsilon>0$ given $\Longrightarrow$ $\exists\delta>0$ s. t. $\|u(t)-u(s)\|\le\varepsilon$ for $\forall t,s\in J$ s. t. $|t-s|<\delta$. Take n s. t. $\frac{1}{n}\le\delta$.

$$\|u_n(t)-u(t)\| < \int_{\mathbb{R}} \varepsilon\psi_n(t-s)\,ds=\varepsilon$$

Ex. 2.1.

1. Since $u(t)\in W^{1,p}(I;X)$, there exists a continuous representative $\tilde{u}(t)\in C(I;X)$. For $r,s\in I$ we have

$$\begin{aligned}\|\tilde{u}(s)-\tilde{u}(r)\|_X &= \left\|\int_r^s \frac{d}{dt}u(t)\,dt\right\|_X \le \left|\int_r^s \left\|\frac{d}{dt}u(t)\right\|_X\,dt\right| \\ &\overset{\text{Hölder}}{\le} |s-r|^{\frac{1}{p'}}\left\|\frac{d}{dt}u(t)\right\|_{L^p(I;X)} \\ &\le |s-r|^{\frac{1}{p'}}\,\|u(t)\|_{W^{1,p}(I;X)},\end{aligned}$$

where $\frac{1}{p'}=1-\frac{1}{p}$.

Since $r,s\in I$ were chosen arbitrary, we have

$$\sup_{\substack{r,s\in I\\ r\neq s}} \frac{\|\tilde{u}(s)-\tilde{u}(r)\|_x}{|s-r|^{1-\frac{1}{p}}} \le \|u(t)\|_{W^{1,p}(I;X)}.$$

Therefore $\tilde{u}(t)$ is α-Hölder for $\alpha=1-\frac{1}{p}$.

3. Denote $\Upsilon=L^1(I;X)$. Choose arbitrary fixed test function $\varphi\in\mathcal{D}(I)$ and define

$$\begin{aligned}\Phi(v(t)) &= \int_I v(t)\varphi'(t)\,dt, \\ \Psi(v(t)) &= -\int_I v(t)\varphi(t)\,dt, \quad v(t)\in\Upsilon.\end{aligned}$$

We see that Φ and Ψ are well–defined linear operators from Υ to X directly by the definition. Moreover, they are continous since

$$\|\Phi(v(t))\|_X \le \int_I \|v(t)\|_X\,|\varphi'(t)|\ \mathrm{dt} \overset{\text{Hölder}}{\le} \|v(t)\|_\Upsilon\,\|\varphi'(t)\|_{L^\infty(I;\mathbb{R})},$$

where $\|\varphi'(t)\|_{L^\infty(I;\mathbb{R})} < \infty$ as φ is a test function and therefore φ' is a continuous compactly supported function. Similar arguments apply for proving the continuity of Ψ. We thus have that $\Phi, \Psi \in \mathcal{L}(\Upsilon, X)$. By the definition of weak derivative we have

$$\Phi(u_n(t)) = \int_I u_n(t)\varphi'(t)\,dt = -\int_I \frac{d}{dt}u_n(t)\varphi(t)\ \mathrm{dt} = \Psi(\frac{d}{dt}u_n(t)).$$

We now have that $\Phi(u_n(t)) \rightharpoonup \Phi(u(t))$ and $\Psi\left(\frac{d}{dt}u_n(t)\right) \rightharpoonup \Psi(g(t))$ in X. By uniqueness of the weak limit it holds that $\Phi(u(t)) = \Psi(g(t))$ and thus

$$\int_I u(t)\varphi'(t)\,dt = \Phi(u(t)) = \Psi(g(t)) = -\int_I g(t)\varphi(t)\,dt,$$

which means that $u(t)$ is weakly differentiable and $\frac{d}{dt}u(t) = g(t)$ since $\varphi \in \mathcal{D}(I)$ was chosen arbitrarily.

5. We can write

$$\int_I \langle v_n(t), u_n(t)\rangle_{X^*,X}\,dt = \int_I \langle v_n(t) - v(t), u_n(t)\rangle_{X^*,X}\,dt + \int_I \langle v(t), u_n(t)\rangle_{X^*,X}\,dt.$$

If we denote $\Upsilon = L^p(I;X)$, $\Upsilon^* = (L^p(I;X))^*$ and denote by $F_n \in \Upsilon^*$ the unique element of dual Υ corresponding to the $v_n(t) \in L^{p'}(I;X^*)$ (and by $F \in \Upsilon^*$ the element corresponding to $v(t) \in L^{p'}(I;X^*)$) by canonical dual mapping, then we have

$$\int_I \langle v(t), u_n(t)\rangle_{X^*,X}\,dt = \langle F, u_n(t)\rangle_{\Upsilon^*,\Upsilon} \overset{n\to\infty}{\longrightarrow} \langle F, u(t)\rangle_{\Upsilon^*,\Upsilon} = \int_I \langle v(t), u(t)\rangle_{X^*,X}\,dt,$$

as $u_n(t) \rightharpoonup u(t)$ in Υ. Moreover, we have that

$$\left|\int_I \langle v_n(t) - v(t), u_n(t)\rangle_{X^*,X}\ \mathrm{dt}\right| \le \int_I \|v_n(t) - v(t)\|_{X^*}\|u_n(t)\|_X\ dt$$
$$\overset{\text{Hölder}}{\le} \|v_n(t) - v(t)\|_{L^{p'}(I;X^*)}\|u_n(t)\|_{L^p(I;X)} \overset{n\to\infty}{\longrightarrow} 0,$$

since the sequence $\{u_n(t)\}$ is weakly convergent in $L^p(I;X)$ and therefeore bounded in $L^p(I;X)$ and $v_n \longrightarrow v$ in $L^{p'}(I;X^*)$. Now it is easily seen that $\int_I \langle v_n(t), u_n(t)\rangle_{X^*,X}\,dt \overset{n\to\infty}{\longrightarrow} \int_I \langle v(t), u(t)\rangle_{X^*,X}\,dt$.

Ex. 2.2. We will show only $L^p(I;L^p(\Omega)) = L^p(I\times\Omega)$ for $p \in [1,\infty)$:

- Let $u = 0$ in $L^p(I\times\Omega) \iff u(t,x) = 0$ for a. e. $(t,x) \in I\times\Omega$. By Fubini we know that this is equivalent to $u(t,x) = 0$ a. e. $x \in \Omega$ for a. e. $t \in I$. Therefore $u(t,\cdot) = 0$ in $L^2(\Omega)$ for a. e. $t \in I$. Moreover $N \subset I\times\Omega$ is zero set $\iff$ for a. e. $t \in I$ the set $N_t = \{x \in \Omega, (t,x) \in N\}$ is zero set. Therefore $u = 0$ in $L^p(I\times\Omega) \iff u(t) = 0$ in $L^2(I;L^p(\Omega))$
- $\|u\|^p_{L^p(I\times\Omega)} = \int_{I\times\Omega}|u(t,x)|^p\,dtdx \overset{\text{Fubini}}{=} \int_I \underbrace{\left(\int_\Omega |u(t,x)|^p\,dx\right)}_{\|u(t,\cdot)\|^p_{L^p(\Omega)}} dt = \|u(t)\|^p_{L^p(I;L^p(\Omega))}$.
- measurability: let $u(t,x) \in L^p(I\times\Omega)$. Then $\exists u_n(t,x) \in C_c^\infty(I\times\Omega)$ such that $u_n(t,x) \to u(t,x)$ in $L^p(I\times\Omega)$. $u_n(t): I \to L^p(\Omega)$ are continuous, therefore strongly measurable.
  Therefore $\int_I \|u_n(t) - u(t)\|_{L^p(\Omega)}\,dt \to 0$ ($u_n(t)$ is Cauchy since $u_n(t,x)$ is Cauchy in $L^p(I\times\Omega)$), hence $u_n(t) \to u$ a. e. in $L^p(I;L^p(\Omega))$

Ex. 3.1.

- Let 1. be true. By part 1: $\frac{d}{dt}\iota u(t) = g(t)$ in I and the function $\iota u(t): I \to W^{-1,2}$ is weakly differentiable. By Lemma 1.3 $\implies$ $\exists\tilde{\tilde{u}}(t) \in AC\left(I, W^{-1,2}\right)$ s. t. $\tilde{\tilde{u}}(t) = \iota u(t)$ a. e. in I. But also (by Thm. 1.12) $\exists\tilde{u}(t) \in C\left(I, L^2\right)$ s. t. $u(t) = \tilde{u}(t)$ a. e. Hence clearly $\tilde{\tilde{u}}(t) = \iota\tilde{u}(t)$ for all $t \in I$ and $\iota u(0) = u_0$. Take $v \in W_0^{1,2}$, $\varphi(t) \in C_c^\infty((-\infty,T))$, clearly $t \mapsto \underbrace{\langle\iota\tilde{u}(t), v\rangle\varphi(t)}_{x(t)}$ is AC.

$$x'(t) = \frac{d}{dt}\left(\langle \iota u(t), v\rangle \varphi(t)\right) = \left\langle \frac{d}{dt}\iota u(t), v\right\rangle \varphi(t) + \langle \iota\tilde{u}(t), v\rangle \varphi'(t) \text{ a. e., write } \underbrace{x(T)}_{=0} - \underbrace{x(0)}_{=-\langle \iota\tilde{u}(0), v\rangle\varphi(0)} = \overbrace{\int_I x'(t)\,dt}^{\int_I \langle g(t), v\rangle \varphi(t) + \langle \iota\tilde{u}(t), v\rangle \varphi'(t)\mathrm{dt}}$$

since $x(t) \in AC$ we have $(u_0, v) = \int_I \langle g(t), v\rangle \varphi(t) + (\tilde{u}(t), v)\varphi'(t)\,dt$, hence 2. holds.

- Let 2. be true: in particular for $\forall \varphi(t) \in C_C^\infty((0,T))$ we have

$$-\int_I (u(t), v)\varphi'(t)\,dt = \int_I \langle g(t), v\rangle \varphi(t)\,dt, \forall v \in W_0^{1,2}$$

$\iff \frac{d}{dt}\iota u(t) = g(t)$, hence we have first part of 1, hence by the above argument (integration by parts): compare with 2. $\implies (\tilde{u}(0), v)\varphi(0) = (u_0, v)\varphi(0)$, where $v \in W_0^{1,2}$ is arbitrary, $W_0^{1,2}$ is dense in L^2 $\implies u_0 = \tilde{u}_0$.

Ex. 3.2.

(1) Let $u \in L^2$. Since f is l-Lipschitz continuous and in particular continuous, $f(u)$ is measurable scalar function as "continuous∘measurable" is a measurable scalar function. For $u, v \in L^2$ we have

$$\begin{aligned}\|f(u) - f(v)\|_2^2 &= \int_\Omega |f(u(x)) - f(v(x))|^2\,dx\\ &\le l^2 \int_\Omega |u(x) - v(x)|^2\,dx = l^2\|u - v\|_2^2,\end{aligned}$$

therefore the mapping $u \in L^2 \mapsto f(u) \in L^2$ is l-Lipschitz.

Now let $u(t) \in L^2(I; L^2)$. In particular, $u(t)$ is strongly measurable and therefore there exist simple functions $u_n(t)$ such that $u_n(t) \to u(t)$ in L^2 for a. e. $t \in I$. Denote by $T_f : L^2 \longrightarrow L^2$ the (nonlinear) operator defined by the formula $T_f(u) = f(u)$. This operator is Lipschitz and therefore continuous due to previous part. Thus $T_f \circ u_n(t) \to T_f \circ u(t) = f(u(t))$ in L^2 for a. e. $t \in I$. But $T_f \circ u_n(t)$ are simple functions (they clearly map their definition domain to finitely many values), which means that $f(u(t))$ is strongly measurable mapping. For $u(t), v(t) \in L^2(I; L^2)$, using previous parts, we have

$$\begin{aligned}\|f(u(t)) - f(v(t))\|_{L^2(I;L^2)}^2 &= \int_I \|f(u(t)) - f(v(t))\|_2^2\,dt\\ &\le l^2 \int_I \|u(t) - v(t)\|_2^2\,\mathrm{dt} = l^2\|u(t) - v(t)\|_{L^2(I;L^2)}^2,\end{aligned}$$

thus the mapping $u(t) \in L^2(I; L^2) \mapsto f(u(t)) \in L^2(I; L^2)$ is l-Lipschitz.

(2) Let $u(t), v(t) \in L^2\left(I; W_0^{1,2}\right)$. Then in particular (for $t \in I$ fixed) we have $f(u(t)), f(v(t)) \in L^2$, as f is Lipschitz and $u(t), v(t) \in L^2$. Fix arbitrary $w \in W_0^{1,2}$, $\|w\|_{1,2} \le 1$. Then we obtain

$$\begin{aligned}|(f(u(t)) - f(v(t)), w)| &\le \int_\Omega |f(u(t)(x)) - f(v(t)(x))|\,|w(x)|\,dx\\ &\le l\int_\Omega |u(t)(x) - v(t)(x)|\,|w(x)|\,dx\\ &\le l\|u(t) - v(t)\|_2\|w\|_2 \le l\|u(t) - v(t)\|_2,\end{aligned}$$

hence

$$\|\iota f(u(t)) - \iota f(v(t))\|_{-1,2} \le l\|u(t) - v(t)\|_2. \tag{5}$$

Using (5), we have

$$\begin{aligned}\|\iota f(u(t)) - \iota f(v(t))\|_{L^2(I;W^{-1,2})}^2 &= \int_I \|\iota f(u(t)) - \iota f(v(t))\|_{-1,2}^2\,dt\\ &\le l^2\int_I \|u(t) - v(t)\|_2^2\,dt \le l^2 \int_I \|u(t) - v(t)\|_{1,2}^2\,dt\\ &= l^2\|u(t) - v(t)\|_{L^2(I;W_0^{1,2})}^2,\end{aligned}$$

which means that the mapping $u(t) \in L^2\left(I; W_0^{1,2}\right) \mapsto \iota f(u(t)) \in L^2\left(I; W^{-1,2}\right)$ is l-Lipschitz.

(3) Let $u, v \in W_0^{1,2}$. Choose $w \in W_0^{1,2}, \|w\|_{1,2} \le 1$ arbitrary fixed. Then

$$\begin{aligned} |\langle \mathcal{A}(u) - \mathcal{A}(v), w\rangle| &\le \int_\Omega |a(\nabla u(x)) - a(\nabla v(x))| \, |\nabla w(x)| \, dx \\ &\le \alpha_1 \int_\Omega |\nabla u(x) - \nabla v(x)| \, |\nabla w(x)| \, dx \\ &\le \alpha_1 \|\nabla u - \nabla v\|_2 \|\nabla w\|_2 \le \alpha_1 \|\nabla u - \nabla v\|_2 \\ &\le \alpha_1 \|u - v\|_{1,2}, \end{aligned}$$

thus $\|\mathcal{A}(u) - \mathcal{A}(v)\|_{-1,2} \le \alpha_1 \|u - v\|_{1,2}$. Therefore the mapping $u \in W_0^{1,2} \mapsto \mathcal{A}(u) \in W^{-1,2}$ is α_1-Lipschitz.

(4) Let $t \in I$ and $u \in W_0^{1,2}$. Fix arbitrary $v \in W^{1,2}, \|v\|_{1,2} \le 1$. Then (we assumed that $a(0) = 0$ and $|\Omega| < \infty$)

$$\begin{aligned} &\left| \int_\Omega a(\nabla u(x)) \cdot \nabla v(x) \, dx + (f(u), v) \right| = \left| \int_\Omega a(\nabla u(x)) \cdot \nabla v(x) \, dx + \int_\Omega f(u(x)) v(x) \, dx \right| \\ &\le \int_\Omega |a(\nabla u(x))| \, |\nabla v(x)| \, dx + \int_\Omega |f(u(x))| \, |v(x)| \, dx \\ &= \int_\Omega |a(\nabla u(x)) - a(o)| \, |\nabla v(x)| \, dx + \int_\Omega |f(u(x))| \, |v(x)| \, dx \\ &\le \alpha_1 \int_\Omega |\nabla u(x)| \, |\nabla v(x)| \, dx + \int_\Omega |f(u(x))| \, |v(x)| \, dx \\ &\le \alpha_1 \|\nabla u\|_2 \|\nabla v\|_2 + \int_\Omega |f(u(x))| \, |v(x)| \, dx \\ &\le \alpha_1 \|\nabla u\|_2 + \int_\Omega |f(u(x)) - f(0)| \, |v(x)| \, dx + |f(0)| \int_\Omega |v(x)| \, dx \\ &\le \alpha_1 \|\nabla u\|_2 + l \int_\Omega |u(x)| \, |v(x)| \, dx + |f(0)| \, |\Omega|^{\frac{1}{2}} \\ &\le \alpha_1 \|\nabla u\|_2 + l \|u\|_2 + |f(0)| \, |\Omega|^{\frac{1}{2}}. \end{aligned}$$

Therefore we have proven that

$$\|-\mathcal{A}(u) - \iota f(u)\|_{-1,2} \le \max\{\alpha_1, l, |f(0)| \, |\Omega|^{\frac{1}{2}}\}(1 + \|u\|_{1,2}). \tag{6}$$

By the inequality (6) it easily follows that

$$\begin{aligned} \|\mathcal{F}(t, u)\|_{-1,2} &= \|-\mathcal{A}(u) - \iota f(u) + h(t)\|_{-1,2} \le \|-\mathcal{A}(u) - \iota f(u)\|_{-1,2} + \|h(t)\|_{-1,2} \\ &\le \max\{1, \alpha_1, l, |f(0)| \, |\Omega|^{\frac{1}{2}}\}(1 + \|u\|_{1,2} + \|h(t)\|_{-1,2}), \end{aligned}$$

which concludes the proof.

Ex. 3.3. Let $W_0^{1,2} \hookrightarrow L^2 \hookrightarrow W^{-1,2}$ be the Gelfand triple, with the embedding $\iota : W_0^{1,2} \to W^{-1,2}$.

(1) By Poincaré inequality, the norms $\|\cdot\| : u \mapsto \|\nabla u\|_2$ and $\|u\|_{1,2}$ are equivalent on $W_0^{1,2}$. Therefore the mapping $((\cdot, \cdot)) : W_0^{1,2} \times W_0^{1,2} \to \mathbb{R}$ defined as $((u, v)) = (\nabla u, \nabla v)$ is a scalar product on $W_0^{1,2}$ and $\left(W_0^{1,2}, ((\cdot, \cdot))\right)$ is a Hilbert space.

(4) Let $f \in W^{-1,2} = \left(W_0^{1,2}\right)^*$ be given. By Riesz theorem, there exists $u_f \in W_0^{1,2}$ so that

$$\langle f, v\rangle = ((u_f, v)) \quad \forall v \in W_0^{1,2}$$

By density of C_c^∞ in $W_0^{1,2}$, there exists a sequence $u_n \in C_c^\infty$ such that $u_n \to u_f$ in $W_0^{1,2}$, hence $\langle f, v\rangle = \lim_{n\to\infty}((u_n, v))$. Now we can write

$$((u_n, v)) = \int_\Omega \nabla u_n(x)\nabla v(x)\,dx = \int_\Omega \Delta u_n(x)\,v(x)\,dx = (-\Delta u_n, v) = \Big\langle \iota \underbrace{(-\Delta u_n)}_{\in C_c^\infty \subset L^2_n}, v\Big\rangle$$

hence $f = \lim_{n\to\infty} \iota f_n$ (limit in $W^{-1,2}$), $f_n \in L^2$

Ex. 3.4.

4. Let $f \in W^{-1,2}$ be given. $W^{-1,2} = \left(W_0^{1,2}\right)^*$, $W_0^{1,2}$ is a Hilbert space with scalar product $((u,v)) = (\nabla u, \nabla v)$. By Riesz theorem we know: $\exists u_f \in W_0^{1,2}$ s. t. $\langle f, v\rangle = ((u_f, v)) = (\nabla u_f, \nabla v)\,\forall v \in W_0^{1,2}$.
   We know: $\exists u_n \in C_c^\infty$ s. t. $u_n \to u_f$ in $W_0^{1,2}$ and therefore $\langle f, v\rangle = \lim_{n\to\infty}(\nabla u_n, \nabla v)$, but due to Green's theorem we have $(\nabla u, \nabla v) = \int_\Omega \nabla u_n \cdot \nabla v dx = -\int_\Omega \underbrace{\Delta u_n}_{=-f_n} v dx = (f_n, v) = \langle \iota f_n, v\rangle$. We have shown: $\langle f, v\rangle = \lim_{n\to\infty}\langle \iota f_n, v\rangle$, i. e., $\iota f_n \rightharpoonup f$ weakly (for $v \in W_0^{1,2}$ fixed). The same holds even strongly: $\|\iota f_n - f\| = \sup_{\substack{v\in W_0^{1,2}\\ \|v\|\le 1}} \langle \iota f_n - f, v\rangle = (\nabla u_n - \nabla u, v) \le \|\nabla u_n - \nabla u\|_2 \to 0$, i. e., the convergence is strong. Therefore $\overline{(\iota L^2)}^{W^{-1,2}} = W^{-1,2}$.
5. it suffices to show that $\iota u = 0$ in $W^{-1,2} \implies u = 0$ in L^2: $\iota u = 0 \iff \langle \iota u, v\rangle = 0$ for $\forall v \in W_0^{1,2}$, i. e., $(u,v) = 0 \forall v \in W_0^{1,2}$ but $W_0^{1,2}$ is dense in $L^2 \implies (u,v) = 0\ \forall v \in L^2 \implies u = 0$ in L^2. Follow by linearity.

Ex. 4.1. Let $u(t) \in L^\infty(I;X) \cap C(I;Z)$. I. e., there exists a measurable subset $N \subseteq I, |N| = 0$ such that for $\forall t \in I \setminus N$ it holds that $u(t) \in X$ and $\|u(t)\|_X \le \|u(t)\|_{L^\infty(I;X)} = M < \infty$. We begin by proving that this holds for every $t \in I$. Choose $t_0 \in N$ arbitrary fixed. As $|N| = 0$ (and in particular N does not contain any nondegenerate interval), there exists $t_n \in I \setminus N$ such that $t_n \to t_0$. For every $n \in \mathbb{N}$ we thus have $u(t_n) \in X$ and $\|u(t_n)\|_X \le M$. As X is reflexive, we can choose a subsequence $\{s_n\} \subseteq \{t_n\}$ such that $u(s_n) \rightharpoonup x_0 \in X$ in X. Since $X \hookrightarrow Z$, we also have $u(s_n) \rightharpoonup x_0$ in Z. But $u(t) \in C(I;Z)$, therefore $u(s_n) \to u(t_0)$ in Z as $s_n \to t_0$. By uniqueness of weak limit it follows that $u(t_0) = x_0 \in X$ and by weak lower semicontinuity of the norm we have $\|u(t_0)\|_X \le \liminf u(s_n) \le M$. We have thus shown that

$$u(t) \in X,$$
$$\|u(t)\|_X \le M, \quad \forall t \in I. \tag{7}$$

Now we proceed to prove that the mapping $t \in I \mapsto u(t) \in X$ is weakly continuous. I. e., we need to show that for arbitrary $\varphi \in X^*$ the mapping $t \in I \mapsto \langle \varphi, u(t)\rangle_{X^*,X}$ is continuous. Fix $\varphi \in X^*$. By (7) we have that for every $t \in I$ it holds that $\langle \varphi, u(t)\rangle_{X^*,X} \in [-\|\varphi\|_{X^*} M, \|\varphi\|_{X^*} M]$. Assume for contradiction that $t \in I \mapsto \langle \varphi, u(t)\rangle_{X^*,X}$ is not continuous. Therefore there exists $t_0 \in I$ and a sequence $t_n \in I$ such that $t_n \to t_0$. But $\langle \varphi, u(t_n)\rangle_{X^*,X} \not\to \langle \varphi, u(t_0)\rangle_{X^*,X} \in [-\|\varphi\|_{X^*} M, \|\varphi\|_{X^*} M]$. Thus there exists a chosen subsequence $\{s_n\} \subseteq \{t_n\}$ such that $\langle \varphi, u(s_n)\rangle_{X^*,X} \to c_0 \in [-\|\varphi\|_{X^*} M, \|\varphi\|_{X^*} M] \setminus \{\langle \varphi, u(t_0)\rangle_{X^*,X}\}$. But by similar reasoning as in first part we know that we can choose a subsequence $\{r_n\} \subseteq \{s_n\}$ such that $u(r_n) \rightharpoonup u(t_0)$ in X. Therefore, in particular $\langle \varphi, u(r_n)\rangle_{X^*,X} \to \langle \varphi, u(t_0)\rangle_{X^*,X}$. This contradicts our assumptions as $\langle \varphi, u(r_n)\rangle_{X^*,X} \to c_0 \ne \langle \varphi, u(t_0)\rangle_{X^*,X}$. The mapping $t \in I \mapsto \langle \varphi, u(t)\rangle_{X^*,X}$ is therefore continuous. It is obvious that $t \in I \mapsto u(t) \in X$ is weakly continuous as $\varphi \in X^*$ was chosen arbitrarily.

Ex. 4.2.

(1) For $u \in W_0^{1,2}$ we have (we know that the eigenvalues λ_j are positive)

$$P_N u = \sum_{j=1}^N (u, w_j) w_j = \sum_{j=1}^N \frac{1}{\lambda_j}(\nabla u, \nabla w_j) w_j = \sum_{j=1}^N ((u, \frac{w_j}{\sqrt{\lambda_j}}))\frac{w_j}{\sqrt{\lambda_j}}.$$

But the set $\left\{\frac{w_1}{\sqrt{\lambda_1}}, \dots, \frac{w_N}{\sqrt{\lambda_N}}\right\}$ is orthonormal in ON v $W_0^{1,2}$ with the scalar product $((\cdot,\cdot))$ as $(\nabla w_j, \nabla w_l) = \delta_{jl}\lambda_j$. We have expressed P_N as (nontrivial) orthonormal projection on span $\left\{\frac{w_1}{\sqrt{\lambda_1}}, \dots, \frac{w_N}{\sqrt{\lambda_N}}\right\}$, therefore the mapping $P_N : W_0^{1,2} \longrightarrow W_0^{1,2}$ is continuous with norm equal to 1 if $W_0^{1,2}$ is considered Hilbert space with the scalar product $((\cdot,\cdot))$.

(2) Add extra assumption $\partial\Omega \in C^2$ for we need to use theorem about elliptic regularity. Application of this theorem to eigenfunctions of laplacian for Dirichlet problem gives that $w_j \in W_0^{1,2} \cap W^{2,2}$ and in particular $\Delta w_j = -\lambda_j w_j$ a. e. in Ω. Now we use second part of this theorem about estimate of $W^{2,2}$ norm and we obtain that for every $v \in W^{2,2}$ it holds that

$$\|v\|_{2,2} \le C_R \left(\|v\|_2 + \|\Delta v\|_2\right), \tag{8}$$

where C_R is independent of v. Let $u \in W_0^{1,2} \cap W^{2,2}$. Since $\frac{\partial u}{\partial x_i} \in W^{1,2}$ and $w_j \in W_0^{1,2}$, we can use integration by parts for Sobolev functions and obtain

$$\int_\Omega \frac{\partial u}{\partial x_i}(x)\frac{\partial w_j}{\partial x_i}(x)\,dx = -\int_\Omega \frac{\partial^2 u}{\partial x_i^2}(x) w_j(x)\,dx.$$

Therefore

$$\begin{aligned}(\nabla u, \nabla w_j) &= \sum_{i=1}^n \int_\Omega \frac{\partial u}{\partial x_i}(x)\frac{\partial w_j}{\partial x_i}(x)\,dx \\ &= -\sum_{i=1}^n \int_\Omega \frac{\partial^2 u}{\partial x_i^2}(x) w_j(x)\,dx \\ &= -(\Delta u, w_j),\end{aligned}$$

by which it follows that

$$\begin{aligned}\Delta P_N u &= \sum_{j=1}^N (u, w_j)\Delta w_j = \sum_{j=1}^N -\lambda_j (u, w_j) w_j \\ &= \sum_{j=1}^N -(\nabla u, \nabla w_j) w_j = \sum_{j=1}^N (\Delta u, w_j) w_j \\ &= P_N \Delta u,\end{aligned}$$

a. e. in Ω. We have thus shown that

$$\Delta P_N u = P_N \Delta u, \tag{9}$$

a. e. in Ω. Combining (8), (9) and the fact that P_N is orthonormal projection $L^2 \longrightarrow L^2$, we obtain that

$$\begin{aligned}\|P_N u\|_{2,2} &\le C_R(\|P_N u\|_2 + \|\Delta P_N u\|_2) = C_R(\|P_N u\|_2 + \|P_N \Delta u\|_2) \\ &\le C_R(\|u\|_2 + \|\Delta u\|_2) \le C_R \|u\|_{2,2}.\end{aligned}$$

Ex. 4.3. By Lemma 2.3 we know that (for this part of the lemma the boundedness of first derivative suffices) $\psi(u) \in W^{1,2}$ for $u \in W^{1,2}$ and that $\nabla\psi(u) = \psi'(u)\nabla u$. Let $u_n \to u$ in $W^{1,2}$. The goal is to prove that $\psi(u_n) \to \psi(u)$ in $W^{1,2}$. Obviously $\psi(u_n) \to \psi(u)$ in L^2 since

$$\int_\Omega |\psi(u_n(x)) - \psi(u(x))|^2\,dx \le \|\psi'\|_\infty^2 \int_\Omega |u_n(x) - u(x)|^2\,dx \to 0,$$

as $\|\psi'\|_\infty < \infty$ and $u_n \to u$ v L^2. It remains to show that

$$\nabla\psi(u_n) \to \nabla\psi(u) \tag{10}$$

in L^2. In first step we will prove (10) with an additional assumption:

$$u_n(x) \to u(x)$$

a. e. in Ω. Then it holds that

$$\|\psi'(u_n)\nabla u_n - \psi'(u)\nabla u\|_2 \le \|\psi'(u_n)\nabla u_n - \psi'(u_n)\nabla u\|_2 + \|\psi'(u_n)\nabla u - \psi'(u)\nabla u\|_2,$$

since

$$\|\psi'(u_n)\nabla u_n - \psi'(u_n)\nabla u\|_2 \le \|\psi'\|_\infty \|\nabla u_n - \nabla u\|_2 \to 0,$$

as $\|\psi'\|_\infty < \infty$ and $\nabla u_n \to \nabla u$ in L^2 and moreover we have that

$$\|\psi'(u_n)\nabla u - \psi'(u)\nabla u\|_2^2 = \int_\Omega |\psi'(u_n(x))\nabla u(x) - \psi'(u(x))\nabla u(x)|^2 \, dx \to 0,$$

because $|\psi'(u_n(x))\nabla u(x) - \psi'(u(x))\nabla u(x)|^2 \to 0$ a. e. in Ω and

$$|\psi'(u_n(x))\nabla u(x) - \psi'(u(x))\nabla u(x)|^2 \leq 4\,\|\psi'\|_\infty^2\,|\nabla u(x)|^2 \in L^1,$$

so we can use Lebesgue dominated convergence theorem. We have thus proved that $\|\psi'(u_n)\nabla u_n - \psi'(u)\nabla u\|_2 \to 0$, which concludes the proof of (10) za with an additional assumption
It remains to prove the general case. Assusme for contradiction that $\nabla\psi(u_n) \not\to \nabla\psi(u)$ in L^2. Therefore there exists a chosen subsequence $\{v_n\} \subseteq \{u_n\}$ such that

$$\|\psi'(v_n)\nabla v_n - \psi'(u)\nabla u\|_2 \to c \in (0, \|\psi'\|_\infty (2\,\|\nabla u\|_2 + 1)].$$

But since $v_n \to u$ in L^2 and $L^2 \hookrightarrow L^1$ (we have $|\Omega| < \infty$), we can choose a subsequence $\{z_n\} \subseteq \{v_n\}$ such that $z_n(x) \to u(x)$ a. e. in Ω. Due to the first step it holds that $\|\nabla\psi(z_n) - \nabla\psi(u)\|_2 \to 0$, which contradicts our assumptions as $\|\nabla\psi(z_n) - \nabla\psi(u)\|_2 \to c \neq 0$. Therefore (10) holds, which remained to prove.

Ex. 5.1. $S(t)$ is a c_0-semigroup in X. The following statements are equivalent:

(1) $S(t) = e^{tA}$, kde $A \in \mathcal{L}(X)$
(2) $S(h) \to I$ v $\mathcal{L}(X)$, $h \to 0^+$.

Remark: (2) ... uniform continuity: $\|S(h) - I\|_{\mathcal{L}(X)} \to 0 \implies \sup_{\|x\|\leq 1} \|S(h)x - x\| \to 0$, $h \to 0^+$ ($\implies$ 3. $S(h)x \to x$, $h \to 0^+$, $\forall x \in X$ fixed).

(1) $\implies$ (2) let $S(t) = e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$ ... absolutely convergent in $\mathcal{L}(X)$ for $\forall t \in \mathbb{C}$, the proof is similar as in the case of $A \in \mathbb{C}^{n\times n}$ since $\mathcal{L}(X)$ is complete and$\|AB\|_{\mathcal{L}(X)} \leq \|A\|_{\mathcal{L}(X)} \|B\|_{\mathcal{L}(X)}$. Moreover $\frac{d}{dt}S(t) = AS(t)$, as a mapping $t \in \mathbb{C} \mapsto S(t) \in \mathcal{L}(X)$ (differentiation temr by term), in particular, $\frac{1}{h}(S(x) - x) \to Ax$, $\forall x \in X$, i. e., we have shown that if $\left(\tilde{A}, D(A)\right)$ is a generator of $S(t) = e^{tA}$ then $D\left(\tilde{A}\right) = X$, $\tilde{A}x = Ax$ for $\forall x \in X$.

(2) $\implies$ (1) let $S(t)$ be a semigroup and $S(h) \to I$, $h \to 0^+$ in $\mathcal{L}(X)$. Denote the generator of $S(t)$ by $(A, D(A))$. We will show that $D(A) = X$, $A \in \mathcal{L}(X) \implies$ the proof is done, define $\tilde{S}(t) = e^{tA}$ in the sense of (1). We know that A generates $\tilde{S}(t)$, due to L. 4.2 we have: $S(t)$ and $\tilde{S}(t)$ are generated by the same operator $\implies S(t) = \tilde{S}(t)$, i. e., $S(t) = e^{tA}$. **Trick:** $S(s) \to I$, $s \to 0^+ \implies \underbrace{\frac{1}{\tau}\int_0^\tau S(s)\,ds}_{B_\tau} \to I$, $\tau \to 0^+ \implies B_\tau = I - \underbrace{(I - B_\tau)}_{\text{small}}$ is an invertible operator: $((I-Q)^{-1} = \sum_{n=0}^{\infty} Q^n$ if $\|Q\|_{\mathcal{L}(X)} < 1)$. $\frac{S(h)-I}{h} B_\tau = \frac{1}{h}\left(S(h)\frac{1}{\tau}\int_0^\tau S(s)\,ds - \int_0^\tau S(s)\,ds\right)$ move $S(h)$ inside the integral in the first term, combine to $S(h+s)$ and obtain $\frac{1}{\tau}\left(\int_\tau^{\tau+h} S(s)\,ds - \frac{1}{h}\int_0^h S(s)\,ds\right)$, $h \to 0^+ : \frac{1}{\tau}(S(\tau) - I)$ since $\frac{1}{h}(S(h) - I)B_\tau \to \frac{1}{\tau}(S(\tau) - I)$ in $\mathcal{L}(X)$, $h \to 0^+$, $\frac{1}{h}(S(h) - I) = \frac{1}{h}(S(h) - I)B_\tau B_\tau^{-1} \to \frac{1}{\tau}(S(\tau) - I)B_\tau^{-1} = A \in \mathcal{L}(X)$. Therefore $\frac{1}{h}(S(h)x - x) \to Ax$ for $\forall x \in X$ fixed, i. e., A is a bounded generator of $S(t)$.

Ex. 5.2. Heat equation (see Chap. 2) $\frac{d}{dt}u - \Delta u = 0$, $t > 0$, $u(0) = u_0$. Th. 2.3 $\implies \forall u_0 \in L^2 \exists! u(t) \in L^2\left(I, W_0^{1,2}\right) \cap C\left(I, L^2\right)$, where $I = [0,T]$ is arbitrary. Denote $S(t)u_0 = u(t)$ (operator which maps the initial condition to solution at the time t).
We claim: $S(t)$ is a c_0-semigroup in L^2.

- $S(t+s) = S(t)S(s)$ ... concatenation of solutions (the solution at time t is considered to be initial condition for solution at time s).
- $S(t) \in \mathcal{L}(X)$ ... linearity of equation,
- $\|S(t)\|_{\mathcal{L}(L^2)} \leq 1$, $\forall t \geq 0$. (Use difference of two solutions $w = u - v$ as a test function $\implies \frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2 + \|\nabla w\|_2^2 = 0 \implies \|w(t)\|_2^2 \leq \|w(0)\|_2^2$)
- Third property: $t \mapsto S(t)u_0 = u(t)$ is continuous (follows from $u(t) \in C\left(I; L^2\right)$).

? generator $(A, D(A))$: auxiliary idea: let $u_0 \in W_0^{1,2}$ (better initial condition) Th. 2.5 $\Longrightarrow$ $u(t) \in L^\infty\left(I, W_0^{1,2}\right) \Longrightarrow$ the mapping $t \mapsto u(t)$ is continuous into the weak topology $W_0^{1,2}$ (see ex. 4.1). On the other hand: by the weak formulation we know: $(u(t), v) = (u_0, v) - \int_0^t \nabla u(s) \cdot \nabla v ds\ \forall t \geq 0,\ v \in W_0^{1,2}$. Moreover: $\left(\frac{S(t)u_0 - u_0}{t}, v\right) = \frac{1}{t}\int_0^t \underbrace{\nabla u(s) \nabla v}_{\text{continuous integrand}} ds$. Take the limit $t \to 0^+$: $\frac{S(t)u_0 - u_0}{h} \to -\tilde{A}u_0$, where $\tilde{A}$ is the weak laplacian from chapter 2: $\left\langle \tilde{A}u, v\right\rangle = \int_\Omega \nabla u \cdot \nabla v dx$. I. e., Δ in some form is the generator. Later we will show that $D(A) = W_0^{1,2} \cap W^{2,2}$, $A : u \mapsto \Delta u \in L^2$.

Ex. 5.3.

(1) Since $(Au)'(t)$ exists, $Au(t+h)$ is well–defined for small h, therefore there exists $\delta > 0$ such that $u(t+h) \in D(A)$ for $0 < |h| < \delta$. For these h we thus have

$$\frac{u(t+h) - u(t)}{h} \in D(A) \longrightarrow u'(t), \quad h \to 0.$$

By linearity of A we have

$$A\left(\frac{u(t+h) - u(t)}{h}\right) = \frac{Au(t+h) - Au(t)}{h} \longrightarrow (Au)'(t), \quad h \to 0.$$

By closedness of operator A it follows that $u'(t) \in D(A)$ and $(Au)'(t) = A(u'(t))$.

(2) Denote $\|u\|_{D(A)} = \|u\|_X + \|Au\|_X$ for $u \in D(A)$. As A is closed, the space $\left(D(A), \|\cdot\|_{D(A)}\right)$ is Banach so we can use the theory of Bochner integral. Since $u(t) \in L^1(I; D(A))$, there exist simple functions $u_n(t) : I \longrightarrow D(A)$ such that $u_n(t) \longrightarrow u(t)$ in $D(A)$ for a. e. $t \in I$. A fortiori we have $u_n(t) \longrightarrow u(t)$ in X for a. e. $t \in I$, since clearly $\left(D(A), \|\cdot\|_{D(A)}\right) \hookrightarrow (X, \|\cdot\|_X)$. Therefore $u(t) : I \longrightarrow X$ is strongly measurable. Moreover $\int_I \|u(t)\|_X\, dt \leq \int_I \|u(t)\|_{D(A)}\, dt < \infty$ and so $u(t) \in L^1(I; X)$ due to Bochner theorem. By linearity of A the functions $Au_n(t) : I \longrightarrow X$ are simple and moreover

$$\|Au_n(t) - Au(t)\|_X \leq \|u_n(t) - u(t)\|_{D(A)} \longrightarrow 0$$

for a. e. $t \in I$. Therefore $Au(t) : I \longrightarrow X$ is strongly measurable. Clearly $\int_I \|Au(t)\|_X\, dt \leq \int_I \|u(t)\|_{D(A)}\, dt < \infty$, thus $Au(t) \in L^1(I; X)$.

We know that

$$\int_I u_n(t)\, dt \in D(A) \longrightarrow \int_I u(t)\, dt,$$

and by linearity of A and the fact that $u_n(t)$ are simple functions we have

$$A\int_I u_n(t)\, dt = \int_I Au_n(t)\, dt \longrightarrow \int_I Au(t)\, dt.$$

By closedness of A it therefore holds that $\int_I u(t)\, dt \in D(A)$ and $A\int_I u(t)\, dt = \int_I Au(t)\, dt$ which concludes the proof.

Ex. 5.4.

(1) $S(t)$ form a c_0-semigroup:
 - (i) $S(t)S(s) = S(t+s)$, $S(0) = I$ ... clear
 - (ii) continuity: we know $f(x+h) \to f(x)$, $h \to 0^+$ in $L^2(\mathbb{R})$. By Ex. 1.4 we know: $S(h)f \to f$, $h \to 0^+$, moreover $\|S(h)\| \leq 1$ independently of $\forall h$.

(2) Fix $t > 0$. Obviously for arbitrary $f \in B_{L^2}$ it holds that

$$\|f(x+t) - f(x)\|_2 \leq \|f(x+t)\|_2 + \|f\|_2 = 2\|f\|_2 \leq 2.$$

Thus

$$\|S(t) - I\|_{\mathcal{L}(X)} \leq 2. \tag{11}$$

To saturate the operator norm it is easily seen that we are looking for functions from unit ball for which $f(x+t)$ is "almost equal" to $-f(x)$. Define

$$f_1(x) = \frac{1}{\sqrt{2t}} \left(\chi_{(0,t)}(x) - \chi_{(-t,0)}(x)\right).$$

Then clearly

$$\|f_1\|_2 = 1$$

and

$$f_1(x+t) = -f_1(x), \text{ for } x \in (-t, 0).$$

Hence

$$\begin{aligned}\|f_1(x+t) - f_1(x)\|_2 &= \frac{1}{\sqrt{2t}} \sqrt{\int_{-2t}^{-t} 1\,dx + \int_{-t}^{0} 4\,dx + \int_{0}^{t} 1\,dx} = \sqrt{\frac{1+4+1}{2}} \\ &= \sqrt{3}.\end{aligned}$$

Define

$$f_2(x) = \frac{1}{\sqrt{4t}} \left(\chi_{(-t,0)\cup(t,2t)}(x) - \chi_{(-2t,-t)\cup(0,t)}(x)\right).$$

Clearly

$$\|f_2\|_2 = 1$$

and

$$f_2(x+t) = -f_2(x), \text{ for } x \in (-2t, t).$$

Thus

$$\begin{aligned}\|f_2(x+t) - f_2(x)\|_2 &= \frac{1}{\sqrt{4t}} \sqrt{\int_{-3t}^{-2t} 1\,dx + \int_{-2t}^{t} 4\,dx + \int_{t}^{2t} 1\,dx} = \sqrt{\frac{1+3\cdot 4+1}{4}} \\ &= \sqrt{\frac{7}{2}}.\end{aligned}$$

In general, define

$$\begin{aligned}f_n(x) = \frac{1}{\sqrt{2tn}} (&\chi_{(-(n-1)t,-(n-2)t)\cup(-(n-3)t,-(n-4)t)\cup\cdots\cup((n-1)t,nt)}(x) \\ &- \chi_{(-nt,-(n-1)t)\cup(-(n-2)t,-(n-3)t)\cup\cdots\cup((n-2)t,(n-1)t)}(x)).\end{aligned}$$

Then obviously

$$\|f_n\|_2 = 1 \tag{12}$$

and

$$f_n(x+t) = -f_n(x), \text{ pro } x \in (-nt, (n-1)t),$$

so

$$\begin{aligned}\|f_n(x+t) - f_n(x)\|_2 &= \frac{1}{\sqrt{2nt}} \sqrt{\int_{-(n+1)t}^{-nt} 1\,dx + \int_{-nt}^{(n-1)t} 4\,dx + \int_{(n-1)t}^{nt} 1\,dx} \\ &= \sqrt{\frac{1+4(2n-1)+1}{2n}} = \sqrt{\frac{4n-1}{n}} \longrightarrow \sqrt{4} = 2.\end{aligned} \tag{13}$$

Therefore by (12) and (13) we see that

$$\|S(t) - I\|_{\mathcal{L}(X)} \geq 2,$$

which togehter with (11) gives the desired result:

$$\|S(t) - I\|_{\mathcal{L}(X)} = 2.$$

(3) Consider the regularization kernels

$$\psi_0(t) = \chi_{(-1,0)}(t),$$
$$\psi_h(t) = \frac{1}{h}\psi_0\left(\frac{t}{h}\right), \quad h > 0.$$

We know that $\mathrm{spt}\psi_h \subseteq [-h, 0]$, and for $f \in L^2$ it holds that $(f * \psi_h)(x) \longrightarrow f(x)$ in L^2. Let $f \in W^{1,2}$. We can assume that f is absolutely continuous and we will obtain, denoting $g = \frac{d}{dx}f \in L^2$, the following:

$$\begin{aligned}
\frac{S(h)f(x) - f(x)}{h} &= \frac{f(x+h) - f(x)}{h} = \frac{\int_x^{x+h} g(s)\,ds}{h} \\
&= \frac{\int_{-h}^0 g(x-s)\,ds}{h} = \frac{\int_{-h}^0 g(x-s)\chi_{(-h,0)}(s)\,ds}{h} \\
&= \frac{\int_{-h}^0 g(x-s)\chi_{(-1,0)}(\frac{s}{h})\,ds}{h} = \frac{\int_{-h}^0 g(x-s)\psi_0(\frac{s}{h})\,\mathrm{ds}}{h} \\
&= \int_{-h}^0 g(x-s)\psi_h(s)\,ds = \int_{\mathbb{R}} g(x-s)\psi_h(s)\,ds \\
&= (g * \psi_h)(x) \longrightarrow g(x) = \frac{d}{dx}f(x)
\end{aligned}$$

in L^2 for $h \to 0_+$, as was to be shown.

(4) Choose an arbitrary, fixed test function $\varphi \in D(\mathbb{R})$. We have

$$\begin{aligned}
\int_{\mathbb{R}} \frac{f(x+h) - f(x)}{h}\varphi(x)\,dx &= \frac{1}{h}\left[\int_{\mathbb{R}} f(x+h)\varphi(x)\,dx - \int_{\mathbb{R}} f(x)\varphi(x)\,dx\right] \\
&= \frac{1}{h}\left[\int_{\mathbb{R}} f(x)\varphi(x-h)\,dx - \int_{\mathbb{R}} f(x)\varphi(x)\,dx\right] \\
&= \int_{\mathbb{R}} f(x)\underbrace{\frac{\varphi(x-h) - \varphi(x)}{h}}_{\to \frac{d}{dt}\varphi = \varphi' \text{ by (3)}}\,dx.
\end{aligned}$$

The proof is completed since the test function φ was chosen arbitrarily and therefore $\frac{d}{dx}f(x) = g(x) \in L^2$ and $f \in W^{1,2}$.

Ex. 6.1.

(1) Let $D(A) = \ell^2$ and let $A \in \mathcal{L}(\ell^2)$. Therefore there exists $C > 0$ such that

$$\|Au\|_2 \le C\,\|u\|_2$$

for every $u \in \ell^2$. In particular by the choice $u = \mathbf{e}_j$ we obtain

$$|\lambda_j| = \sqrt{\sum_{k=1}^{\infty} \lambda_k^2 u_k^2} = \|A\mathbf{e}_j\|_2 \le C\,\|\mathbf{e}_j\|_2 = C,$$

and thus the sequence $\{\lambda_j\}$ is bounded.

On the other hand, let $\{\lambda_j\}$ be bounded. Then clearly for every $u \in \ell^2$ we have

$$\|Au\|_2 = \sqrt{\sum_{k=1}^{\infty} \lambda_k^2 u_k^2} \le \|\{\lambda_j\}\|_\infty \sqrt{\sum_{k=1}^{\infty} u_k^2} = \|\{\lambda_j\}\|_\infty \|u\|_2 < \infty.$$

Hence $D(A) = \ell^2$ and $A \in \mathcal{L}(\ell^2)$, the linearity is evident.

(2) Let $u^n \in D(A) \longrightarrow u$ and $Au^n \longrightarrow v$ in ℓ^2. We will choose a rapidly convergent subsequence which converges pointwise. I. e., we will choose $\{w^n\} \subseteq \{u^n\}$ such that

$$w_j^n \longrightarrow u_j, \quad n \to \infty \tag{14}$$

and

$$(Aw^n)_j = \lambda_j w_j^n \longrightarrow v_j, \quad n \to \infty \tag{15}$$

for every $j \in \mathbb{N}$. By (14) we thus have that

$$\lambda_j w_j^n \longrightarrow \lambda_j u_j, \quad n \to \infty,$$

which along with (15) gives the result

$$\lambda_j u_j = v_j,$$

for every $j \in \mathbb{N}$, therefore $u \in D(A)$ and $Au = v$. Therefore the operator A is closed.

The operator is moreover densely defined as it is sufficient to consider finitely supported sequences, which are dense in ℓ^p for $p \in [1, \infty)$ (in particular for $p = 2$). Finitely supported sequences are clearly contained in the definition domain of A and since they are dense in ℓ^2, a fortiori $D(A)$ is dense in ℓ^2.

(3) Moreover, let $\lambda_j \leq \omega$ and $\lambda > \omega$. The goal is to prove that $\lambda \in \rho(A)$, i. e., the operator $\lambda I - A : D(A) \longrightarrow \ell^2$ is bijective. Let $\lambda u - Au = \lambda v - Av$ for $u, v \in D(A)$. Therefore for every $j \in \mathbb{N}$ we have that

$$\lambda u_j - \lambda_j u_j = \lambda v_j - \lambda_j v_j,$$

which means that $u_j = v_j$ as $\lambda - \lambda_j \neq 0$. Hence $u = v$, i. e., the operator $\lambda I - A$ is injective.

Let $v \in \ell^2$ be given. We are looking for $u \in D(A)$ such that $\lambda u - Au = v$. The goal is to prove that for every $j \in \mathbb{N}$ it holds that

$$\lambda u_j - \lambda_j u_j = v_j,$$

thus the only candidate for the argument of A is

$$u = \left\{ \frac{v_j}{\lambda - \lambda_j} \right\}. \tag{16}$$

It remains to show that this u is in the definition domain of A. Clearly $u \in \ell^2$, since

$$\sum_{j=1}^{\infty} \frac{v_j^2}{(\lambda - \lambda_j)^2} \leq \frac{1}{(\lambda - \omega)^2} \sum_{j=1}^{\infty} v_j^2 = \frac{1}{(\lambda - \omega)^2} \|v\|_2^2 < \infty.$$

Moreover

$$\sum_{j=1}^{\infty} \frac{\lambda_j^2 v_j^2}{(\lambda - \lambda_j)^2} \leq \max\left\{1, \frac{\omega^2}{(\lambda - \omega)^2}\right\} \sum_{j=1}^{\infty} v_j^2 < \infty,$$

where the inequality $\frac{\lambda_j^2}{(\lambda-\lambda_j)^2} \leq \max\left\{1, \frac{\omega^2}{(\lambda-\omega)^2}\right\}$ can be easily obtained by analysis of the function $f(x) = \frac{x^2}{(\lambda-x)^2}$ on the interval $(-\infty, \omega]$. Therfore $u \in D(A)$, hence the operator$\lambda I - A$ is onto.

From this it may be concluded that $\lambda \in \rho(A)$ and by (16) we know that $R(\lambda, A)v = \left\{\frac{v_j}{\lambda-\lambda_j}\right\}$ for $\lambda > \omega$ and $v \in \ell^2$.

(4) Since the convergence in ℓ^2 in particular means that we are able to choose a pointwise convergent subsequence, it is easy to guess a candidate for $S(t)$:

$$S(t)u = \left\{ \mathrm{e}^{\lambda_j t} u_j \right\} \tag{17}$$

for $u \in \ell^2$ and $t \in [0, \infty)$. Operator defined in this way is clearly linear and bounded on ℓ^2, since

$$\sum_{j=1}^{\infty} \mathrm{e}^{2\lambda_j t} u_j^2 \leq \mathrm{e}^{2\omega t} \sum_{j=1}^{\infty} u_j^2,$$

therefore $S(t) \in \mathcal{L}(\ell^2)$. Moreover, it is evident that $S(0) = \mathrm{I}$ and $S(t+s) = S(t)S(s)$ for $s, t \geq 0$. Fix $u \in \ell^2$. We want to show that

$$S(t)u \longrightarrow u, \quad t \to 0_+. \tag{18}$$

Clearly

$$\mathrm{e}^{\lambda_j t} u_j \longrightarrow u_j, \quad t \to 0_+ \tag{19}$$

for every $j \in \mathbb{N}$ and for $0 < t < 1$ we have

$$(\mathrm{e}^{\lambda_j t} u_j - u_j)^2 \leq 2(\mathrm{e}^{2\lambda_j t} u_j^2 + u_j^2) \leq 2(\mathrm{e}^{2\omega t} u_j^2 + u_j^2) \leq 2(\mathrm{e}^{2|\omega|} u_j^2 + u_j^2), \tag{20}$$

where $\left\{2(\mathrm{e}^{2|\omega|}u_j^2 + u_j^2)\right\} \in \ell^1$, hence by (19), (20) and Lebesgue dominated convergence theorem (using the arihmetic measure on $\mathbb{N}$) the equation (18) holds. Thus $S(t)$ is a c_0-semigroup.

It suffices to show that $(A, D(A))$ is a generator of this semigroup. Let $u \in D(A)$. Clearly

$$\frac{\mathrm{e}^{\lambda_j h} u_j - u_j}{h} \longrightarrow \lambda_j u_j, \quad h \to 0_+ \tag{21}$$

for every $j \in \mathbb{N}$. Since

$$\frac{\mathrm{e}^{\lambda_j h} - 1}{h} \leq \frac{\mathrm{e}^{\omega h} - 1}{h} \longrightarrow \omega,\ h \to 0_+,$$

there exists $h_0 > 0$ (independent on j) such that for everry $h \in (0, h_0)$ it holds that

$$\frac{\mathrm{e}^{\lambda_j h} - 1}{h} \leq 2\,|\omega|\,.$$

Using the inequality $\mathrm{e}^x \geq 1 + x$ we have

$$\frac{1 - \mathrm{e}^{\lambda_j h}}{h} \leq \frac{-\lambda_j h}{h} \leq |\lambda_j|\,.$$

Combining these steps together we obtain that for every $j \in \mathbb{N}$ and every $h \in (0, h_0)$ it holds that

$$\left|\frac{\mathrm{e}^{\lambda_j h} - 1}{h}\right| \leq \max\left\{|\omega|\,, |\lambda_j|\right\} \leq |\omega| + |\lambda_j|\,. \tag{22}$$

Using (22) we get that for $h \in (0, h_0)$ the following holds

$$\left|\frac{\mathrm{e}^{\lambda_j h} - 1}{h} u_j - \lambda_j u_j\right|^2 \leq 2\left(\left|\frac{\mathrm{e}^{\lambda_j h} - 1}{h}\right|^2 u_j^2 + \lambda_j^2 u_j^2\right) \leq 2\left(2\left(\omega^2 + \lambda_j^2\right) u_j^2 + \lambda_j^2 u_j^2\right),$$

where $\left\{5\lambda_j^2 u_j^2 + 4\omega^2 u_j^2\right\} \in \ell^1$ as $u \in D(A)$, which together with (21) and Lebesgue dominated convergence theorem yields

$$\frac{S(h)u - u}{h} \longrightarrow Au, \quad h \to 0_+.$$

It remains to show that $D(A) = \left\{u \in \ell^2; \lim\limits_{h \to 0_+} \frac{S(h)u - u}{h} \in \ell^2\right\}$.

We already know that $D(A) \subseteq \left\{u \in \ell^2, \lim\limits_{h \to 0_+} \frac{S(h)u - u}{h} \in \ell^2\right\}$. To prove the opposite inclusion, consider

$$u \in \left\{v \in \ell^2, \lim_{h \to 0_+} \frac{S(h)v - v}{h} \in \ell^2\right\}$$

and let $\lim\limits_{h \to 0_+} \frac{S(h)u - u}{h} = v \in \ell^2$. In particular, for $h_n = \frac{1}{n}$ it holds that $\frac{S(h_n)u - u}{h_n} \longrightarrow v$. Choose pointwise converging subsequence, i. e., choose a subsequence $\{\tau_n\} \subseteq \{h_n\}$ such that for every $j \in \mathbb{N}$ we have

$$\frac{\mathrm{e}^{\tau_n \lambda_j} u_j - u_j}{\tau_n} \longrightarrow v_j, \quad n \to \infty,$$

but

$$\frac{\mathrm{e}^{\tau_n \lambda_j} u_j - u_j}{\tau_n} \longrightarrow \lambda_j u_j, \quad n \to \infty,$$

hence $v_j = \lambda_j u_j$. I. e., $\{\lambda_j u_j\} = v \in \ell^2$, therefore $u \in D(A)$.

We have thus proven (two c_0-semigroups generated by the same operator are identic, see L. 4.2) that $S(t)$ defined by (17) is clearly a c_0-semigroup generated by the operator $(A, D(A))$.

Ex. 6.2. Let $S(t)$ be a c_0-semigroup generated by $(A, D(A))$ such that $\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$. Denote $\widetilde{S}(t) = e^{-\omega t} S(t)$. Then:

(1) $\widetilde{S}(t)$ is a semigroup, $\left\|\widetilde{S}(t)\right\|_{\mathcal{L}(X)} \leq M \ \forall t \geq 0$

(2) If $\left(\widetilde{A}, D\left(\widetilde{A}\right)\right)$ generates $\widetilde{S}(t)$, then $D(A) = D\left(\widetilde{A}\right)$, $\widetilde{A} = A - \omega I$

(3) $\rho\left(\widetilde{A}\right) = \rho(A) - \omega$, $R\left(\lambda, \widetilde{A}\right) = R(\lambda + \omega, A) \ \forall \lambda \in \rho\left(\widetilde{A}\right)$.

Proof: 1., 3. ... easy (expand)

2. it suffices to show: $x \in D(A) \implies x \in D\left(\widetilde{A}\right)$ and $\widetilde{A}x = Ax - \omega x$:

$\frac{1}{h}\left[\widetilde{S}(h)x - x\right] = \frac{1}{h}\left[e^{-\omega h} S(h)x - x \pm S(h)x\right] = \underbrace{\frac{e^{-\omega h}-1}{h}}_{\to -\omega} \underbrace{S(h)x}_{\to x} + \underbrace{\frac{1}{h}[S(h)x - x]}_{\to Ax}$

Ex. 6.3. Heat semigroup $\mathcal{S}(t): L^2 \to L^2$, $u_0 \mapsto u(t)$ ... weak solution (in the sense of chap. 2) of the heat equation (HE) $\quad \frac{d}{dt}u - \Delta u = 0$, $u|_{t=0} = 0$, $u|_{\partial\Omega} = 0$, $x \in \Omega$ in the space $L^2\left(I, W_0^{1,2}\right) \cap C\left(I, L^2\right)$. What is the generator of $\mathcal{S}(t)$?

Recall: we know: $\exists \lambda_j > 0$, $w_j \in W_0^{1,2}$ ... eigenfunctions of laplacian, i. e., $-\Delta w_j = \lambda_j w_j$ weakly in Ω: $(\nabla w_j, \nabla v) = \lambda_j (w_j, v) \ \forall v \in W_0^{1,2}$. Moreover $\{w_j\}$ form a complete ON basis of L^2, i. e., $\forall u \in L^2$ can be expressed as $u = \sum_{j=1}^{\infty} u_j w_j$, where $u_j = (u, w_j)$; Parseval: $\|u\|_{L^2}^2 = \sum_j u_j^2$.

Statement 6.3.2:

(a) $u_0 \in W_0^{1,2} \iff \sum_j \lambda_j u_j^2 < \infty$, in particular, $\sum_j \lambda_j u_j = \|\nabla u_0\|_2^2$.

(b) $u_0 \in W_0^{1,2} \cap W^{2,2} \iff \sum_j \lambda_j^2 u_j^2 < \infty$, in particular, $\sum_j \lambda_j u_j = \|\Delta u_0\|_2^2$

Proof of (b):

" $\Longleftarrow$ " let $u_0 \in W_0^{1,2}$, $\sum \lambda_j^2 u_j^2 < \infty$, where $u_j = (u_0, w_j) \overset{?}{\Longrightarrow} u_0 \in W^{2,2}$. Denote $u_N = P_N u_0 = \sum_{j=1}^{N} u_j w_j$: we know: $u_N \to u_0$ v L^2. Moreover: $-\Delta u_N = \sum_{j=1}^{N} u_j (-\Delta u_j) = \underbrace{\sum_{j=1}^{N} \lambda_j u_j w_j}_{\text{denote by } z_N}$ weakly. We know: $\|-\Delta u_N\|_2^2 = \|z_N\|_2^2 = \left\|\sum_{j=1}^{N} \lambda_j^2 u_j w_j\right\|_2^2 = \sum_{j=1}^{N} \lambda_j^2 w_j^2 \leq \sum_{j=1}^{\infty} \lambda_j^2 u_j^2 = c < \infty$. We will use elliptic regularity: $\|u_N\|_{2,2} \leq C_R \|z_N\|_2 \leq C_R\sqrt{c}$, where C_R depends only on Ω. Therefore $\{u_N\} \subset W^{2,2}$ is bounded $\implies \exists$subsequence $\widetilde{u}_N \rightharpoonup \widetilde{u} \in W_0^{1,2}$, therefore $\widetilde{u}_N \rightharpoonup \widetilde{u} \in L^2$

" $\Longrightarrow$ " let $u_0 \in W_0^{1,2} \cap W^{2,2} \overset{?}{\Longrightarrow} \sum_{j=1}^{\infty} \lambda_j^2 u_j^2 < \infty$ and is equal to $\|\Delta u_0\|_2^2 = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}$ (weak derivative).

Recall: Gauss–Green theorem: $(\nabla u, \nabla w) = (-\Delta u, w)$, $\forall u \in W^{2,2}$, $w \in W_0^{1,2}$.

Proof: $0 = \int_{\partial\Omega} w (\nabla u \cdot n)\, dS = \int_\Omega \underbrace{\operatorname{div}(w \nabla u)}_{\nabla w \cdot \nabla u + w \Delta u} dx = \int_\Omega \nabla w \cdot \nabla u dx + \int_\Omega w \Delta u dx$, ... u, w is smooth and follow by density in given spaces.

Apply to $u = u_0 \in W^{2,2}$, $w = w_j \in W_0^{1,2}$ and obtain: $(\nabla u_0, \nabla w_j) = (-\Delta u_0, w_j) = z_j$. $LS = \lambda_j (u_0, w_j)$ since $-\Delta w_j = \lambda_j w_j$ weakly, $u_0 \in W_0^{1,2}$ "test function". Parseval: $\sum z_j^2 = \|-\Delta u_0\|_2^2$, at the same time we know that $\sum z_j^2 = \sum \lambda_j^2 u_j^2$.

Statement 6.3.1: $u_0 \in L^2$ are given, $u_0 = \sum_j u_j w_j \implies \sum_j e^{-\lambda_j t} u_j w_j$ is a weak solution of (HE).

Corollary: the generator of heat semigroup is $\underbrace{\Delta}_{=\sum_{j=1}^{n} \frac{\partial^2}{\partial x_i^2}}$ with the definition domain $W_0^{1,2} \cap W^{2,2}$.

Proof of corollary: the statement implies that $\left(\mathcal{S}(t), L^2\right) \leftrightarrow \left(\widetilde{\mathcal{S}}(t), \ell^2\right)$, where $\widetilde{\mathcal{S}}(t)$ is a semigroup generated by the operator $\widetilde{A}: \{u_j\} \mapsto \{\lambda_j u_j\}$, we know (see ex. 6.1 and ex. 6.3): $D\left(\widetilde{A}\right) = \left\{\sum_j \lambda_j^2 u_j^2 < \infty\right\} \implies D(A) = W_0^{1,2} \cap W^{2,2}$.

Proof of statement 6.3.1: we will show (i) $u(t) \in L^2\left(I, W_0^{1,2}\right)$, (ii) $\frac{d}{dt}(u(t), v) + (\nabla u(t), \nabla w) = 0$ weakly in $[0,T]$ for $\forall w \in W_0^{1,2}$ fixed.

(i) $\|\nabla u(t)\|_2^2 \overset{6.3.2,\text{ a)}}{=} \sum_j \lambda_j e^{-2\lambda_j t} u_j^2$ since $u(t) = \sum_j e^{-\lambda_j t} u_j w_j$.

$$\int_0^T \|\nabla u(t)\|_2^2 \, dt = \int_0^T \sum_j \lambda_j e^{-2\lambda_j t} u_j^2 dt = \sum_j \underbrace{\int_0^T \lambda_j e^{-2\lambda_j t} u_j^2 dt}_{\left[\frac{e^{-2\lambda_j t}}{-2}\right]_0^T u_j^2} \leq \frac{1}{2}\|u_0\|_2^2$$

Remark: from chapter 2 we know: $\frac{1}{2}\|u(T)\|_2^2 + \int_0^T \|\nabla u(t)\|_2^2 \, dt = \frac{1}{2}\|u_0\|_2^2$.

(ii) due to the density it suffices to consider $w = w_k$, where $k = 1, 2, \ldots$ is arbitrary fixed. Then $(u(t), w) = \left(\sum_j e^{-\lambda_j t} u_j w_j, w_k\right) \overset{\text{ON}}{=} e^{-\lambda_k t} u_k$. $(\nabla u(t), \nabla w) = \left(\nabla \sum_j e^{-\lambda_j t} u_j w_j, \nabla w_k\right) = \sum_j e^{-\lambda_j t} u_j \underbrace{(\nabla w_j, \nabla w_k)}_{= \lambda_k, k=1; 0 \text{ otherwise } \ldots \text{ OG v } W_0^{1,2}}$ since the sum converges also in $W_0^{1,2}$ $\ldots = \lambda_k e^{-\lambda_k t} u_k$.

Ex. 6.4.

(1) Let $\varepsilon > 0$ be given and $(t_0, x_0) \in [0, \infty) \times X$. Since the mapping $t \in [0, \infty) \mapsto S(t)x_0$ is continuous, there exists $\delta > 0$ such that

$$\|S(t)x_0 - S(t_0)x_0\|_X \leq \frac{\varepsilon}{2} \tag{23}$$

for every $t \in (t_0 - \delta, t_0 + \delta) \cap [0, \infty)$. Moreover we know that there exists constats $M \geq 1$ and $\omega \geq 0$ such that

$$\|S(t)x - S(t)x_0\|_X \leq M\mathrm{e}^{\omega t}\|x - x_0\|_X \tag{24}$$

for every $t \geq 0$. For $(t, x) \in ((t_0 - \delta, t_0 + \delta) \cap [0, \infty)) \times B\left(x_0, \frac{\varepsilon}{2M\mathrm{e}^{\omega(t_0+\delta)}}\right)$ we obtain by (23) and (24) the following:

$$\begin{aligned}\|S(t)x - S(t_0)x_0\|_X &\leq \|S(t)x - S(t)x_0\|_X + \|S(t)x_0 - S(t_0)x_0\|_X \\ &\leq M\mathrm{e}^{\omega t}\|x - x_0\|_X + \frac{\varepsilon}{2} \leq M\mathrm{e}^{\omega(t_0+\delta)}\|x - x_0\|_X + \frac{\varepsilon}{2} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

thus the mapping $(t, x) \in [0, \infty) \times X \mapsto S(t)x$ is continuous.

(2) Fix $T > 0$ and $x \in X$. Fix an arbitrary $t_0 \in [0, T]$. We know that

$$\tilde{S}(t_0 + h)x \longrightarrow \tilde{S}(t_0)x, \quad h \to 0,$$

and by the first part we know that the mapping $(t, x) \in [0, \infty) \times X \mapsto S(t)x$ is continuous, hence

$$S(T - (t_0 + h))(\tilde{S}(t_0 + h)x) \longrightarrow S(T - t_0)\tilde{S}(t_0)x, \quad h \to 0,$$

and the mapping $y(t)$ is continuous in t_0, which was chosen arbitrarily (in the case of $t_0 = 0$ or $t_0 = T$ it suffices to consider limit from the right or from the left, respectively).

(3) Denote by $(A, D(A))$ the common generator of $S(t)$ and $\tilde{S}(t)$. Let $x \in D(A)$ and $t \in (0, T)$. The goal is to prove that $y'(t) = 0$. At first, compute the derivative from the right:

$$\frac{\tilde{S}(t + h)x - \tilde{S}(t)x}{h} = \tilde{S}(t)\left(\frac{\tilde{S}(h)x - x}{h}\right) \longrightarrow \tilde{S}(t)Ax, \quad h \to 0_+,$$

as $\frac{\tilde{S}(h)x - x}{h} \longrightarrow Ax$ for $h \to 0_+$ $(x \in D(A))$ and $S(t) \in \mathcal{L}(X)$. Using the first part:

$$\begin{aligned}&\frac{S(T - (t + h))\tilde{S}(t + h)x - S(T - (t + h))\tilde{S}(t)x}{h} \\ &= S(T - (t + h))\left(\frac{\tilde{S}(t + h)x - \tilde{S}(t)x}{h}\right) \longrightarrow S(T - t)\tilde{S}(t)Ax, \quad h \to 0_+.\end{aligned} \tag{25}$$

By properties of generator ($x \in D(A)$) we know that $\tilde{S}(t)x \in D(A)$, so

$$\frac{\tilde{S}(t)x - S(h)\tilde{S}(t)x}{h} \longrightarrow -A\tilde{S}(t)x, \quad h \to 0_+,$$

and once more by using the continuity from the first part we get

$$\begin{aligned}
&\frac{S(T-(t+h))\tilde{S}(t)x - S(T-t)\tilde{S}(t)x}{h} \\
&= S(T-(t+h))\left(\frac{\tilde{S}(t)x - S(h)\tilde{S}(t)x}{h}\right) \longrightarrow -S(T-t)A\tilde{S}(t)x, \quad h \to 0_+.
\end{aligned} \tag{26}$$

By combining (25) and (26) we thus obtain

$$\begin{aligned}
&\frac{y(t+h)-y(t)}{h} \\
&= \frac{S(T-(t+h))\tilde{S}(t+h)x - S(T-(t+h))\tilde{S}(t)x}{h} \\
&+ \frac{S(T-(t+h))\tilde{S}(t)x - S(T-t)\tilde{S}(t)x}{h} \\
&\longrightarrow S(T-t)\tilde{S}(t)Ax - S(T-t)A\tilde{S}(t)x = 0, \quad h \to 0_+,
\end{aligned}$$

since $\tilde{S}(t)Ax = A\tilde{S}(t)x$ from the properties of generator it follows that $y'(t_+) = 0$.

For the derivative from the left let us write

$$\begin{aligned}
&\frac{y(t-h)-y(t)}{-h} \\
&= \frac{S(T-(t-h))\tilde{S}(t-h)x - S(T-(t-h))\tilde{S}(t)x}{-h} \\
&+ \frac{S(T-(t-h))\tilde{S}(t)x - S(T-t)\tilde{S}(t)x}{-h} \\
&= S(T-(t-h))\left(\tilde{S}(t-h)\frac{\tilde{S}(h)x - x}{h}\right) \\
&- S(T-t)\left(\frac{S(h)\tilde{S}(t)x - \tilde{S}(t)x}{h}\right)
\end{aligned}$$

and then it is sufficient to apply similar reasoning.

(4) Fix an arbitrary $T > 0$ and $x \in D(A)$. By the second part we know that the mapping $y(t)$ is continuous on $[0, T]$. By the third part we know that $y'(t) = 0$ for $t \in (0, T)$. Therefore $y(t)$ is constant on $[0, T]$, which in particular means that

$$\tilde{S}(T)x = y(T) = y(0) = S(T)x.$$

As $T > 0$ and $x \in D(A)$ were chosen arbitrarily, we have derived that

$$\tilde{S}(t)x = S(t)x$$

for every $t \geq 0$ (for $t = 0$ it holds trivially) and $x \in D(A)$. Since the generator of c_0-semigroup is densely defined, it follows by the continuity of operators $S(t), \tilde{S}(t)$ (t fixed), that

$$\tilde{S}(t)x = S(t)x$$

for $x \in X$, which concludes the proof.

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Appendices

Properties of weak convergence

Let $u_n \rightharpoonup u$ in X Then the following statements hold:

(1) $Lu_n \rightharpoonup Lu$ in Y if $L : X \to Y$ is linear continuous

(2) $u_n \rightharpoonup u$ in Y if $X \hookrightarrow Y$

(a) $u_n \to u$ in Y, if $X \hookrightarrow\hookrightarrow Y$

Proof:

(1) goal: $\langle g, Lu_n\rangle_{Y^*,Y} \to \langle g, Lu\rangle_{Y^*,Y}$ for $\forall g \in Y*$ fixed. Observe: mapping $f : u \mapsto \langle g, Lu\rangle_{Y^*,Y} \in X^*, X \to \mathbb{R}$ (is usually denoted $\underbrace{L^*g}_{=f}$: $L^* : Y^* \to X^*$).

Conclusion follows as $u_n \rightharpoonup u$, hence $\langle f, u_n\rangle \to \langle f, u\rangle$.

(2) special case of 1: $X \hookrightarrow Y$ means: $Id : X \to Y$ is continuous.

(3) $u_n \rightharpoonup u$ in $X \implies \|u_n\|_X$ is bounded (Banach-Steinhaus), hence $X \hookrightarrow\hookrightarrow Y$ implies $\exists$ subsequence $\underbrace{\tilde{u}_n \to u}_{\implies \tilde{u}_n \rightharpoonup u \text{ in } Y}$ in Y, on the other hand $\tilde{u}_n \rightharpoonup u$ in Y, hence $u = \tilde{u}$, because the weak limit is unique (Hahn-Banach).

We need more: $u_n \to u$ without taking subsequences. By contradiction:

$u_n \nrightarrow u$, hence $\exists\varepsilon > 0\exists$subsequence $\tilde{\tilde{u}}_n$ s. t. $\left\|\tilde{\tilde{u}}_n - u\right\|_Y \geq \varepsilon\forall n$, we can assume $\tilde{\tilde{u}}_n \overset{(\rightharpoonup)}{\to} \tilde{\tilde{u}}$ in Y, clearly $\left\|\tilde{\tilde{u}} - u\right\|_Y \geq \varepsilon$, but on the other hand $\tilde{\tilde{u}}_n \rightharpoonup u$ in Y ... contradiction.

Time derivative of integral over time-dependent domain

Let $e(t,x)$ be smooth.

$$\frac{d}{dt}\int_{B(x_0,\tau-t)} e(t,x)\,dx = \int_{B(x_0,\tau-t)} \overset{(1)}{\partial_t e(t,x)}\,dx - \int_{\partial B(x_0,\tau-t)} \overset{(2)}{e(t,x)}\,dS_x$$

Proof: WLOG $x_0 = 0$, **TRICK:** substitution $x = (\tau - t)\,y,\ dx = (\tau-t)^n\,dy,\ y \in B(0,1)$, where n is dimension.

$$\int_{B(0,\tau-t)} e(t,x)\,dx \overset{\text{subst.}}{=} \int_{B(0,1)} e(t,(\tau-t)\,y)\cdot(\tau-t)^n\,dx$$

$$LHS = \int_{B(0,1)} \partial_t e(t,(\tau-t)\,y)\cdot(\tau-t)^n + e(t,(\tau-t)\,y)\cdot n(\tau-t)^{n-1} + \nabla_x e(t,(\tau-t)\,y)\cdot(\tau-t)^n\,y\,dy$$

$$\text{subst. } y \to x = \int_{B(0,\tau-t)} \underbrace{\partial_t e(t,x)}_{(1)} + \nabla_x e(t,x)\cdot\frac{x}{\tau-t} + \frac{n}{\tau-t}e(t,x)\,dx$$

The two remaining terms result in (2) since

$$\begin{aligned}
\int_{\partial B(x_0,\tau-t)} e(t,x)\,dS_x &= \int_{\partial B(x_0,\tau-t)} \left[e(t,x)\,\frac{x}{\tau-t}\right]\cdot\nu\,dS_x \\
&= \int_{B(x_0,\tau-x)} \operatorname{div}_x\left[e(t,x)\underbrace{\frac{x}{\tau-t}}_{\nu\ \ldots\ \text{outer normal}}\right]dx \\
(\text{Gauss}) &= \int_{\partial B(x_0,\tau-t)} e(t,x)\underbrace{|\nu|^2}_{1}\,dS_x
\end{aligned}$$

Regularity of the heat equation

Consider the following problem:

$$\begin{aligned} \frac{d}{dt}u - \Delta u &= F(t) \\ u(0) &= u_0 \end{aligned}$$

Th. 2.3: $u_0(t) \in L^2$, $F(t) \in L^2\left(I, W^{-1,2}\right) \implies \exists$w. s. $u(t) \in L^\infty\left(I, L^2\right) \cap L^2\left(I, W_0^{1,2}\right)$, $\frac{d}{dt}u(t) \in L^2\left(I, W^{-1,2}\right)$

Th. 2.5: $u_0(t) \in W_0^{1,2}$, $F(t) \in L^2\left(I, L^2\right) \implies u(t) \in L^\infty\left(I, W_0^{1,2}\right) \cap L^2\left(I, W^{2,2}\right)$, $\frac{d}{dt}u(t) \in L^2\left(I, L^2\right)$.

Th. R: let $u_0(t) \in W^{2m+1,2}$, $\frac{d^k}{dt^k}F(t) \in L^2\left(I, W^{2m-2k,2}\right)$, $k = 0, \ldots, m$ and let the compatibility conditions $(C_0)\ldots(C_m)$ hold:

$$\begin{aligned} (C_0) \quad g_0 &= u_0 \in W_0^{1,2} \\ (C_1) \quad g_1 &= F(0) + \Delta g_0 \in W_0^{1,2} \\ &\vdots \\ (C_m) \quad g_m &= \frac{d^{m-1}}{dt^{m-1}}F(0) + \Delta g_{m-1} \in W_0^{1,2} \end{aligned}$$

then $\frac{d^k}{dt^k}u(t) \in L^2\left(I, W^{2m+2-2k,2}\right)$, $k = 0, \ldots, m+1$.

Remarks concerning the proof of Theorem R.

- $m = 0$: assumptions: $u_0 \in W_0^{1,2}$, $F(t) \in L^2\left(I, L^2\right)$, (C_0): $u_0 \in W_0^{1,2}$. Conlusion: for $k = 0$: $u(t) \in L^2\left(I, W^{2,2}\right)$, $k = 1$: $\frac{d}{dt}u(t) \in L^2\left(I, \underbrace{W^{0,2}}_{=L^2}\right)$, i. e., Th. 2.5, already proven.
- $m = 1$: differentiate the equation with respect to time and use Th. 2.5 for the derivative. Denote $v = \frac{d}{dt}u$:

$$\begin{aligned} \frac{d^2}{dt^2}u + \frac{d}{dt}(-\Delta u) &= \frac{d}{dt}F(t) \\ \frac{d}{dt}v - \Delta v &= \frac{d}{dt}F(t) \\ v(0) &= \frac{d}{dt}u(0) \overset{\text{the equation}}{=} F(0) + \Delta u(0) \end{aligned}$$

  assumptions: $(k = 1)$ $\frac{d}{dt}F(t) \in L^2\left(I, \underbrace{W^{0,2}}_{L^2}\right)$, (C_1): $F(0) + \Delta u(0) = v(0) \in W_0^{1,2}$, due to Th. 2.5.

  we have: $v(t) \in L^2\left(I, W^{2,2}\right)$, $\frac{d}{dt}v(t) \in L^2\left(I, L^2\right)$. We want: $\frac{d^k}{dt^k}u \in L^2\left(I, W^{4-2k,2}\right)$ for $k = 0, 1, 2$. For $k = 1, 2$ it is already done due to Th. 2.5. Remains to show for $k = 0$: $u(t) \in L^2\left(I, W^{4,2}\right)$ ... obtain from the equation for $u(t)$ and th regularity of laplacian: $-\Delta u(t) = \underbrace{F(t) - \frac{d}{dt}u(t)}_{\in W^{2,2}}$. I. e., $\Delta u(t) \in W^{2,2} \overset{\text{regularity of laplacian}}{\Longrightarrow} u(t) \in W^{4,2}$ and $\|u(t)\|_{4,2} \le c\left(\left\|F(t) - \frac{d}{dt}u(t)\right\|_{2,2}\right)$.
- for $m \in \mathbb{N}$ general follow by induction, the induction step is always Th. 2.5. Rigorous proof: approximation (e. g., Galerkin).

Remark. Why do we need the compatibility conditions? We can not prescribe the initial and boundary conditions completely arbitrarily. The compatibility conditions mean correspondence of boundary and initial conditions for $t = 0$, $x \in \partial\Omega$, where these two conditions overlap.
Other equations:

(1) Linear elliptic operator $\mathcal{A}(u) = -\mathrm{div}\,(a(x)\nabla u) + b(x)\nabla u + c(x)\,u \ldots$ the same techniques work for reasonable (e. g. smooth) $a(x), b(x), c(x) \ldots$ See Evans: PDE.
(2) Nonlinear operator: more difficult to prove, but similar technique also works.
(3) Other terms in equation: $\partial_t u - \Delta u + f(u) = h(t)$ rewrite as: $\partial_t u - \Delta u = \underbrace{h(t) - f(u)}_{=F(t)}$, here additional regularity of $f(\cdot)$ is needed.

Niemytskii operators: $u(x) \mapsto f(u(x))$:$L^p(\Omega) \to L^p(\Omega)$, $f(z) : \mathbb{R} \to \mathbb{R}$ continuous, typically $|f(z)| \le c(1 + |z|^\rho)$, $p > 0$.

- measurability $f(u(x)) \ldots$ easy to show by Lebesgue theory,
- integrability: $f(u(x))$: $u(x) \in L^p(\Omega) \implies f(u(x)) \in L^q(\Omega)$ for $\forall q$ s. t. $\rho q \le p$,
proof:$\int_\Omega |f(u)|^q\,dx \le \int_\Omega (c(1 + |u|^\rho))^q\,dx \le c_1 \int_\Omega 1 + |u|^{\rho q}\,dx < \infty$ (estimate $(a+b)^q \le c_q(a^q + b^q)$ for $\forall a, b, q \ge 0$),
- actually we even have that the mapping $u \mapsto f(u)$ is continuous $L^p \to L^q$ as long as the assumptions above hold. (See web pages `http://www.karlin.mff.cuni.cz/~prazak/vyuka/Pdr2/`.)