Ex 2.1. Let X be reflexive, separable.

1. Let $p \in (1, \infty)$. Show that any $u(t) \in W^{1,p}(I;X)$ has a α-Hölder continuous representative, with $\alpha = 1 - 1/p$.

2. Show that $W^{1,\infty}(I;X) = C^{0,1}(I;X)$ (the space of Lipschitz functions), in the sense of representative.

3. Let $u_n(t)$ be weakly differentiable, and let $u_n(t) \rightharpoonup u(t)$, $\frac{d}{dt}u_n(t) \rightharpoonup g(t)$ (weakly) in $L^1(I;X)$. Then $u(t)$ is weakly differentiable, with $\frac{d}{dt}u(t) = g(t)$.

4. Let $u_n(t)$ are bounded in $L^p(I;Y)$, $\frac{d}{dt}u_n(t)$ are bounded in $L^q(I;Z)$, where p, $q \in (1,\infty)$ and Y, Z are reflexive, separable. Then there is a subsequence so that $\tilde{u}_n(t) \rightharpoonup u(t)$, $\frac{d}{dt}\tilde{u}_n(t) \rightharpoonup g(t)$ in the respective spaces, and $\frac{d}{dt}u(t) = g(t)$.

* **Ex 2.2.**

1. Prove that $L^p(I;L^p(\Omega)) = L^p(I \times \Omega)$ if $p \in [1,\infty)$, but $L^\infty(I;L^\infty(\Omega)) \subsetneq L^\infty(I \times \Omega)$.

2. Prove that if $u_n \rightharpoonup u$ in $L^p(I;L^q(\Omega))$, and $\Omega \subset \mathbb{R}^n$ is open, bounded, then

$$\int_{I\times\Omega} u_n(t,x)\psi(t,x)\,dtdx \to \int_{I\times\Omega} u(t,x)\psi(t,x)\,dtdx$$

   for any (say) bounded, measurable function $\psi(t,x)$.

Ex 2.3. Let H be a Hilbert space. Show that H is uniformly convex. Show directly that if $u_n \rightharpoonup u$ and $\|u_n\|_H \to \|u\|_H$, then $u_n \to u$.

* **Ex 2.4.** Let $u_n(t)$ be bounded in $L^p(I;X)$, where $p \in (1,\infty]$, and X be reflexive, separable. Prove that there is a weakly convergent ($*$-weak if $p = \infty$) subsequence, using only Theorem 1.9 and separability of $L^{p'}(I;X^*)$.

Ex 2.5. Let $u_n(t) \rightharpoonup u(t)$ in $L^p(I;X)$, $v_n(t) \to v(t)$ in $L^{p'}(I;X^*)$, where p, p' are Hölder conjugate. Prove that $\int_I \langle u_n(t), v_n(t)\rangle_{X,X^*}\,dt \to \int_I \langle u(t), v(t)\rangle_{X,X^*}\,dt$.

HINTS.

Ex 2.1.

1. By Lemma 1.5, there is a continuous representative $\tilde{u}(t)$ such that $\tilde{u}(t_1) - \tilde{u}(t_2) = \int_{t_1}^{t_2} \frac{d}{dt} u(s)\, ds$. Estimate the integral using the Hölder inequality.
2. Inclusion $\subset$ is as above. For the converse, note that Lipschitz function is absolutely continuous, and its derivative is L^∞, cf. Theorem 1.5.
3. Explain (in detail), that weak convergence is enough to pass in the definition of the weak derivative.
4. Use Eberlein-Šmulian and the previous problem.

Ex 2.3.

1. Use $\|\frac{x+y}{2}\|_H^2 + \|\frac{x-y}{2}\|_H^2 = \|x\|_H^2 + \|y\|_H^2$.
2. Write $\|u - u_n\|_H^2 = \|u\|_H^2 - 2(u, u_n)_H + \|u_n\|_H^2$.

Ex 2.4. Let $v_n(t)$ be a countable dense set in $L^{p'}(I; X^*)$ ...

Ex 2.5. Add and subtract $\langle u_n(t), v(t)\rangle$.