***Ex 3.1.** Let $u(t) \in L^2(I; W_0^{1,2})$, $g(t) \in L^2(I; W^{-1,2})$ and $u_0 \in L^2$. Then the following are equivalent:

(i) $\frac{d}{dt}u(t) = g(t)$ and $u(0) = u_0$ (in the sense of representative)

(ii) for any $v \in W_0^{1,2}$, $\varphi \in C_c^\infty((-\infty, T))$ one has

$$-\int_I (u(t), v)\varphi'(t)\, dt = \int_I \langle g(t), v\rangle\, \varphi(t)\, dt + (u_0, v)\varphi(0)$$

Ex 3.2. Recall the notation and assumptions from Chapter 2: let $f(z) : \mathbb{R} \to \mathbb{R}$, $a(\xi) : \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz continuous. Let the operators $\mathcal{A} : W_0^{1,2} \to W^{-1,2}$ and $\mathcal{F} : I \times W_0^{1,2} \to W^{-1,2}$ be defined as

$$\langle \mathcal{A}(u), v\rangle = \int_\Omega a(\nabla u(x)) \cdot \nabla v(x)\, dx$$
$$\mathcal{F}(t, u) = -\mathcal{A}(u) - \iota f(u) + h(t)$$

where $h(t) \in L^2(I; W^{-1,2})$ is a fixed function.

1. Prove that $u \mapsto f(u)$ is Lipschitz as operator $L^2 \to L^2$, and also $u(t) \mapsto f(u(t))$ is Lipschitz as operator $L^2(I; L^2) \to L^2(I; L^2)$). N.B. Do not forget to verify that $f(u)$ and $f(u(t))$ are *measurable* in the appropriate sense.
2. Deduce that also $u(t) \mapsto \iota f(u(t))$ is Lipschitz as operator $L^2(I; W_0^{1,2}) \to L^2(I; W^{-1,2})$.
3. Prove that $u \to \mathcal{A}(u)$ is Lipschitz continuous as operator $W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$.
4. Show that $\|\mathcal{F}(t, u)\|_{-1,2} \le c(1 + \|u\|_{1,2} + \|h(t)\|_{-1,2})$ with some constant only depending on the nonlinearities $a(\cdot)$ and $f(\cdot)$.

***Ex 3.3.** Let $W_0^{1,2} \hookrightarrow L^2 \hookrightarrow W^{-1,2}$ be the Gelfand triple, with the embedding $\iota : W_0^{1,2} \to W^{-1,2}$.

1. Observe that due to the Poincaré inequality, $W_0^{1,2}$ is a Hilbert space with the scalar product $(\!(u, v)\!) = \int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx$.
2. By Riesz theorem, any $f \in W^{-1,2}$ can be represented by some $u_f \in W_0^{1,2}$ so that

$$\langle f, v\rangle = (\!(u_f, v)\!) \qquad \forall v \in W_0^{1,2}$$

3. Show that by a Green formula $(\!(u, v)\!) = (-\Delta u, v)$ for any $v \in W_0^{1,2}$ and $u \in C_c^\infty$.
4. Combine that above with the density of C_c^∞ in $W_0^{1,2}$ to show that for any $f \in W^{-1,2}$ there exist smooth functions u_n such that $\iota u_n \to f$.
5. Finally, show that $\iota : W_0^{1,2} \to W^{-1,2}$ is *injective*.

These are the reasons why no symbol "ι" is normally employed, and $(\cdot, \cdot)_{L^2}$ is seen simply as a generalization of $\langle \cdot, \cdot\rangle_{W^{-1,2}, W_0^{1,2}}$ without further notational ado.