1. $$\int \ln\left(1+\sqrt[4]{x}\right)dx$$

2. $$\int \frac{\operatorname{tg}^3 x+\operatorname{tg}^2 x+\operatorname{tg} x+1}{\operatorname{tg}^3 x+\operatorname{tg}^2 x+3\operatorname{tg} x-5}\,dx$$

3. $$\lim_{n\to\infty} x_n; \qquad x_0=3, \quad x_{n+1}=\frac{1}{2}\left(x_n+\frac{3}{x_n}\right)$$

4. $$\lim_{x\to 0}\frac{\exp(-x^2)-2\cos x+\sqrt[3]{1+x^3}}{x^2\sin x}$$

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1. $$\int \sin^2 x\cos^2 x\,dx$$

2. $$\int \frac{\ln x}{(\ln x^2-1)\left((\ln x)^2+\ln x+1\right)}\cdot\frac{dx}{x}$$

3. $$\lim_{n\to\infty} x_n; \qquad x_0=0, \quad x_{n+1}=1+\frac{x_n}{1+x_n}$$

4. $$\lim_{x\to\infty} x\left(\frac{\pi}{4}-\operatorname{arctg}\sqrt{\frac{x}{x+1}}\right)$$

(1) $\int \ln\left(1+x^{\frac{1}{4}}\right) dx \;\Big|\;$ per-partes
$u' = 1, \quad u = x$
$v = \ln(\ ); \quad v' = \frac{1}{4} \frac{x^{-3/4}}{1+x^{1/4}}$

$$= x \cdot \ln\left(1+x^{\frac{1}{4}}\right) - \frac{1}{4}\int \frac{x^{1/4}}{1+x^{1/4}}\, dx$$

$$I = \left|\begin{array}{l} \text{subst.} \\ x = y^4 \\ dx = 4y^3 \end{array}\right| = \int \frac{y^4}{1+y}\, dy = \left|\begin{array}{l} \text{subst} \\ 1+y = z \\ y = z-1 \end{array}\right| \quad dy = dz$$

$z = 1+\sqrt[4]{x}$

$$\int \frac{(z-1)^4}{z}\, dz = \int \frac{1}{z} - 4 + 6z - 4z^2 + z^3\, dz = \dots$$

celkem $(x+1)\ln\left(1+x^{1/4}\right) - 4\left(1+x^{1/4}\right) + 3\left(1+x^{1/4}\right)^2 - \frac{4}{3}\left(1+x^{1/4}\right)^3 + \frac{1}{4}\left(1+x^{1/4}\right)^4$

$x > 0.$

(2) $\int \frac{tg^3x + tg^2x + tg\,x + 1}{tg^3x + tg^2x + 3\,tg\,x - 5}\, dx \;\Big|\; t = tg\,x, \quad dx = \frac{dt}{1+t^2}$

$$= \int \frac{t+1}{t^3+t^2+3t-5}\, dt\ ; \qquad \frac{t+1}{(t-1)(t^2+2t+5)} = \frac{1}{4}\cdot\frac{1}{t-1} - \frac{1}{4}\frac{t-1}{t^2+2t+5}$$

$$\frac{t-1}{t^2+2t+5} = \frac{1}{2}\,\frac{2t+2}{t^2+2t+5} - 2\cdot\frac{1}{t+2t+5}$$

$$\left(\ln\left(t^2+2t+5\right)\right)' \qquad \frac{1}{(t+1)^2+4} = \frac{1}{4}\cdot\frac{1}{\left(\frac{t+1}{2}\right)^2+1} = \frac{1}{4}\left(\operatorname{arctg}\frac{t+1}{2}\right)'$$

celkem: $\frac{1}{4}\ln|\mathrm{tg}\,x-1| - \frac{1}{8}\ln(\mathrm{tg}^2 x + 2\,\mathrm{tg}\,x + 5) + \frac{1}{4}\,\mathrm{arctg}\left(\frac{\mathrm{tg}\,x+1}{2}\right)$;

-- intervaly neobsahující $\frac{\pi}{2}+k\pi$; $\frac{\pi}{4}+k\pi$.

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(3) numericky: $x_0 = 3$

$x_1 = 1.75$

$x_2 = 1.73$

$\vdots$

nechť $x_n \to \hat{x}$; $\hat{x} = \left(\hat{x} + \frac{3}{\hat{x}}\right)\frac{1}{2}$

$(\hat{x})^2 = 3$; $\Rightarrow \hat{x} = \pm\sqrt{3}$!

Tedy: pokud limita existuje, je buď $\sqrt{3}$, nebo $-\sqrt{3}$.

indukcí: (a) $x_n > 0 \quad \forall n$

dokážeme (b) $x_n > \sqrt{3} \quad \forall n$

postupně: (c) $x_{n+1} < x_n \quad \forall n$

tedy $\{x_n\}$ je monotónní, omezená $\Rightarrow$ má limitu;

musí $x_n \to \sqrt{3}$.

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(4) Taylor: $x^2(\sin x) = x^2(x + o(x)) = x^3 + o(x^3)$

$\exp(y) = 1 + y + \frac{1}{2}y^2 + o(y^2)$

$\exp(-x^2) = 1 - x^2 + \frac{1}{2}x^4 + o(x^4) = 1 - x^2 + o(x^3)$

$-2\cos x = -2\left(1 - \frac{x^2}{2} + o(x^3)\right) = -2 + x^2 + o(x^3)$

$\sqrt[3]{1+y} = 1 + \frac{1}{3}y + o(y)$

$\sqrt[3]{1+x^3} = 1 + \frac{1}{3}x^3 + o(x^3)$

$$f(x) = \frac{1 - x^2 - 2 + x^2 + 1 + \frac{1}{3}x^3 + o(x^3)}{x^3 + o(x^3)} = \frac{\frac{1}{3}x^3 + o(x^3)}{x^3 + o(x^3)} \to \frac{1}{3}.$$

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(1)
$$\sin^2 y = \frac{1}{2}(1 - \cos 2y)$$
$$\cos^2 y = \frac{1}{2}(1 + \cos 2y)$$

$$f(x) = \frac{1}{4}(1 - \cos^2 2x) = \frac{1}{4}\left(1 - \frac{1}{2}(1 + \cos 4x)\right)$$
$$= \frac{1}{8} + \frac{1}{8}\cos 4x \; ;$$

$$\int \sin^2 x \cos^2 x \, dx = \frac{1}{8}x + \frac{1}{32}\sin 4x \; ; \quad x \in \mathbb{R}$$

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(2)
$$\int \frac{\ln x}{(2\ln x - 1)((\ln x)^2 + \ln x + 1)} \cdot \frac{dx}{x} = \left| \begin{array}{l} \text{subst.} \\ y = \ln x \\ dy = \frac{dx}{x} \end{array} \right.$$

$$= \int \frac{y}{(2y-1)(y^2+y+1)} \, dy$$

$$\frac{y}{(2y-1)(y^2+y+1)} = \frac{2}{7} \cdot \frac{1}{2y-1} - \frac{1}{7}\frac{y-2}{y^2+y+1} \; ;$$

$$\frac{y-2}{y^2+y+1} = \frac{1}{2}\frac{2y+1}{y^2+y+1} - \frac{5}{2} \cdot \frac{1}{y^2+y+1} \; ;$$

$$\frac{1}{y^2+y+1} = \frac{1}{(y+\frac{1}{2})^2 + \frac{3}{4}} = \frac{4}{3} \cdot \frac{1}{\left(\frac{2y+1}{\sqrt{3}}\right)^2 + 1} = \frac{2}{\sqrt{3}} \operatorname{arctg}\left(\frac{2y+1}{\sqrt{3}}\right) ;$$

celkem: $\frac{1}{7}\ln|2\ln x - 1| - \frac{1}{14}\ln(\ln^2 x + \ln x + 1) + \frac{5}{21}\sqrt{3}\operatorname{arctg}\left(\frac{2\ln x + 1}{\sqrt{3}}\right)$

$x \in (0, \sqrt{e}) \; ; \; x \in (\sqrt{e}, \infty).$

(3) numericky: $x_1 = 1$

$x_2 = 1.5$

$x_3 = 1.6$

$\vdots$

? nechť $x_n \to a$. potom: $a = 1 + \frac{a}{1+a}$

$a = \frac{1}{2}(1 \pm \sqrt{5})$.

indukcí: $x_n > 0 \ \forall n$ ... hypotéza: $x_n \to a_2 = \frac{1}{2}(1+\sqrt{5})$

indukcí: $x_n < a_2 \ \forall n$

$x_{n+1} > x_n \ \forall n$

$\{x_n\}$ je omezená, rostoucí $\Rightarrow$ limita existuje.

(4) $\lim\limits_{x\to\infty} x\left(\frac{\pi}{4} - \operatorname{arctg}\sqrt{\frac{x}{x+1}}\right)$ $\qquad x = \frac{1}{y}$; $y \to 0+$

$= \lim\limits_{y\to 0+} \dfrac{\frac{\pi}{4} - \operatorname{arctg}(1+y)^{-\frac{1}{2}}}{y}$, $\qquad \operatorname{arctg} 1 = \frac{\pi}{4}$

l'Hospital „$\frac{0}{0}$"

$$\frac{-\left[\operatorname{arctg}(1+y)^{-\frac{1}{2}}\right]'}{y'} = \underbrace{\frac{1}{1+\frac{1}{1+y}}}_{\to \frac{1}{2}} \cdot \left(-\frac{1}{2}\right) \underbrace{(1+y)^{-\frac{3}{2}}}_{\to 1} \to \underline{\underline{+\frac{1}{4}}}$$