A1) $F(a)=\int_0^{+\infty} \underbrace{\frac{\operatorname{arctg} x}{1+x^a}}_{f(a,x)} dx$.

nebot $|\operatorname{arctg} y| \le \frac{\pi}{2}$

(i) konvergence: $x\in(0,1)$: $|f(a,x)| \le \frac{\pi}{2} \in L(0,1)$ pro $\forall a\in\mathbb{R}$

$x\ge 1$: $f(a,x) \sim 1$, $x\to+\infty$, $a\le 0$

$\sim \frac{1}{x^a}$, " , $a>0$

Závěr: $\boxed{F(a)<+\infty \iff a>1.}$

(ii) spojitost uvnitř $a\in(1,+\infty)$: $f(a,\cdot)$ měřitelné ($\Leftarrow$ spoj.)

$f(\cdot,x)$ spojité,

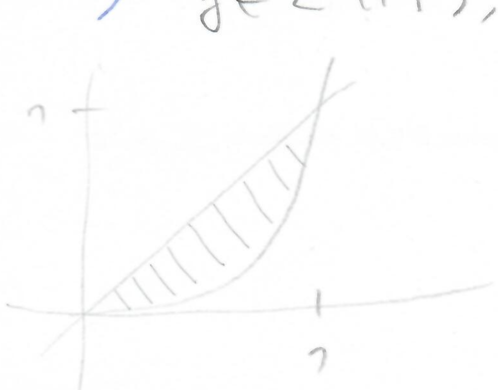
majoranta: polož $\tilde{I}=(\tilde{a},+\infty)$, $\tilde{a}>1$ pevné.

$$\Rightarrow |f(a,x)| \le \left.\begin{cases} \frac{\pi}{2}, & x\in(0,1) \\ \frac{\pi}{2x^{\tilde{a}}}, & x>1\end{cases}\right\} =: g(x) \quad \forall a\in \tilde{I}$$

$\Rightarrow F(a)$ spojitá v $(\tilde{a},+\infty)$, $\forall \tilde{a}>1$ pevné

$\Rightarrow$ $\boxed{F(a) \text{ spojitá v } (1,+\infty)}$

A2) $f\in L^2(\Pi)$, neboť $f\ge 0$, Π uzavřená



$$\int_\Pi f(x,y)\,dx\,dy = \int_0^1 \left(\underbrace{\int_{x^2}^{x} xy^2\, dy}\right)dx$$

$$x\cdot\left[\frac{y^3}{3}\right]_{y=x^2}^{y=x} = \frac{1}{3}(x^4-x^7)$$

$$=\frac{1}{3}\left[\frac{x^5}{5}-\frac{x^8}{8}\right]_{x=0}^{x=1} = \dots = \boxed{\frac{1}{40}}$$

B1) $F(a) = \int_0^{+\infty} \ln(1+ax^2)\cdot \ln(1+x^2)\cdot \frac{dx}{x^2}$

(i) konvergence: $a<0 \dots \Rightarrow 1+ax^2<0$, tj. $f(a,x)$ nemá smysl pro x velké.

$a \geq 0$: v $0+$: $\frac{\ln(1+x^2)}{x^2} \to 1$, tj. $f(a,x) \to 0$, $x\to 0+$

$\Rightarrow f(a,\cdot) \in L(0,1)$.

v $+\infty$: $f(a,x) = \underbrace{\frac{1}{x^{3/2}}}_{\in L(1,+\infty)} \cdot \underbrace{\frac{\ln(1+ax^2)\cdot \ln(1+x^2)}{x^{1/2}}}_{\to 0,\ x\to+\infty}$

tedy ≤ 1 ; x velké

(ii) spojitost vůči $a \geq 0$: $f(a,\cdot)$ měřitelné ($\Leftarrow$ spoj.)

$f(\cdot,x)$ spojité ... jasné.

majoranta: $\sup_{a\geq 0} f(a,x) = \lim_{a\to+\infty} f(a,x) = +\infty$

polož $\tilde{I} = [0,K]$; $K \geq 1$ pevné.

$a \in \tilde{I}$: $|f(a,x)| = \ln(1+ax^2)\cdot \ln(1+x^2)\frac{1}{x^2} \leq \frac{\ln^2(1+Kx^2)}{x^2}$

$\in L(0,+\infty)$ podobně jako výše.

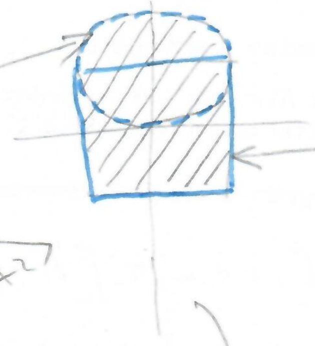
$\Rightarrow F(a)$ spojitá v $[0,K]$, $\forall K>0$

$\Rightarrow F(a)$ spojitá v $[0,+\infty)$

B2) $f \in L(M) \Leftarrow f$ omezená, M omezená, uzavřená.

$x^2+y^2 \le 2y$

$\boxed{x^2+(y-1)^2 \le 1}$



$|x| \le 1, \ |y| \le 1$

$$\int_M f(x,y)\,dx\,dy = \int_{-1}^{1} \Big(\underbrace{\int_{-1}^{1+\sqrt{1-x^2}} (y-1)^3\,dy}_{I(x)} \Big)\,dx =$$

$$I(x) = \left[\frac{1}{4}(y-1)^4\right]_{y=-1}^{y=1+\sqrt{1-x^2}} = \frac{1}{4}\left((1-x^2)^2 - 16\right)$$

$$= \int_{-1}^{1} \frac{x^4}{4} - \frac{x^2}{2} - \frac{15}{4}\,dx = \cdots = \boxed{-\frac{116}{15}}$$

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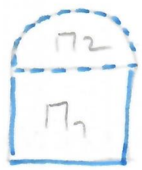
jinak: $M_1 = \{|x| \le 1, |y| \le 1\}$

$M_2 = \{x^2+(y-1)^2 \le 1\} \cap \{y > 1\}$.



$$I_1 = \iint_M f\,dx\,dy = \int_{-1}^{1} \Big(\int_{-1}^{1} (y-1)^3\,dx \Big)\,dy = 2\int_{-1}^{1} (y-1)^3\,dy = -8$$

$$I_2 = \int_1^2 \Big(\underbrace{\int_{-\sqrt{1-(y-1)^2}}^{+\sqrt{1-(y-1)^2}} (y-1)^3\,dx}_{2(y-1)^3\sqrt{1-(y-1)^2}} \Big)\,dy = \int_1^2 2(y-1)^3\sqrt{1-(y-1)^2}\,dy \quad \left|\ \begin{array}{l} y = 1+\sin t \\ dy = \cos t \\ t \in [0, \frac{\pi}{2}] \end{array}\right.$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^3 t \cdot \cos^2 t\,dt = \cdots = \frac{4}{15}$$