2. Trajectory and streamline

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Problem 1.
Consider flow that is described in a lagrangian way using equations

$$x = Xe^{\alpha t}, \quad y = Ye^{-\alpha t}, \quad z = Z,$$

where X, Y and Z are the initial position of a fluid parcel and $\alpha > 0$.

- Find its trajectory.
- Find and plot its streamlines.
- Decide whether the flow is stationary.

Solution:
To find the trajectory, it is necessary to eliminate time in the lagrangian description of the flow. By substituting the factor $e^{\alpha t}$ from function x to y, we get:

$$x = \frac{XY}{y}, \quad z = Z.$$

Trajectories are therefore hyperbolas lying in the xy plane.

Streamlines are defined as curves, whose tangent line is (in every point) the velocity vector. To find the streamlines, we therefore need to find how the velocity looks like. In lagrangian description, the velocity is defined as

$$u(X,Y,Z,t) = \left(\frac{\partial x}{\partial t}\right)_{(X,Y,Z)} = \alpha X e^{\alpha t},$$
$$v(X,Y,Z,t) = \left(\frac{\partial y}{\partial t}\right)_{(X,Y,Z)} = -\alpha Y e^{-\alpha t},$$
$$w(X,Y,Z,t) = \left(\frac{\partial z}{\partial t}\right)_{(X,Y,Z)} = 0.$$

To switch from the description to eulerian, we have to express individual components as functions of x, y and z. Therefore,

$$u(x,y,z,t) = \alpha x, \quad v(x,y,z,t) = -\alpha y, \quad w(x,y,z,t) = 0.$$

Plotting these velocities in (infinite number of) different points, we get the streamlines, see Fig. 1.

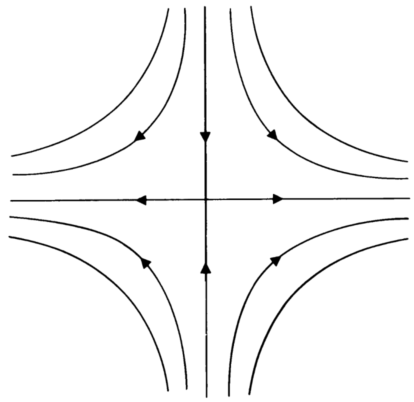

Figure 1: Streamlines for exercise 1.

These streamlines are again hyperbolas, as well as the trajectories. This is not a coincidence – because the eulerian velocities are not dependent on time, the flow is stationary and the trajectories thus coincide with streamlines.

Problem 2.
Consider the nonstationary flow

$$u = u_0, \quad v = kt, \quad , w = 0,$$

where u_0 and k are positive constants. Derive how the streamlines look like and find the trajectories.

Solution:
Streamlines $x = x(s)$, $y = y(s)$ a $z = z(s)$ are in every time t described by equations

$$\frac{\frac{\mathrm{d}x(s)}{\mathrm{d}s}}{u} = \frac{\frac{\mathrm{d}y(s)}{\mathrm{d}s}}{v} = \frac{\frac{\mathrm{d}z(s)}{\mathrm{d}s}}{w}$$

(their direction should be the same as the ratio of the velocity components). We thus want to compute

$$\frac{\frac{\mathrm{d}y(s)}{\mathrm{d}s}}{\frac{\mathrm{d}x(s)}{\mathrm{d}s}} = \frac{v}{u}.$$

Using the chain rule, the left-hand side is $\mathrm{d}y/\,\mathrm{d}x$, whereas the right-hand side is, after substituting the velocity components, kt/u_0. By integration of the remaining differential equations, we get:

$$y = \frac{kt}{u_0}x + const.$$

The second equality in the definition implies:

$$z = const.$$

Streamlines are therefore at every time instant straight lines in the xy plane.

As for the trajectories, it is possible to obtain them from integrals of velocity in lagrangian description (constant initial positions of the parcels) with respect to time:

$$\left(\frac{\partial x}{\partial t}\right)_{\mathbf{X}} = u_0, \quad \left(\frac{\partial y}{\partial t}\right)_{\mathbf{X}} = kt, \quad \left(\frac{\partial z}{\partial t}\right)_{\mathbf{X}} = 0,$$

therefore

$$x = u_0 t + F_1(\mathbf{X}), \quad y = \frac{1}{2}kt^2 + F_2(\mathbf{X}), \quad z = F_3(\mathbf{X}).$$

Functions F_1, F_2 and F_3 are specified so that at time $t = 0$, it is $\mathbf{x} = \mathbf{X}$. The motion of the parcels can be therefore described by equations

$$x = u_0 t + X, \quad y = \frac{1}{2}kt^2 + Y, \quad z = Z.$$

Finally, we get the trajectory from the elimination of time:

$$y = \frac{1}{2}k\left(\frac{X - x}{u_0}\right)^2 + Y$$

Although the streamlines at a fixed time are straight-lines, the parcels are moving along parabolas.