3. Material derivative

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Problem 1.
Consider flow that is described in a lagrangian way using equations

$$x = Xe^{\alpha t}, \quad y = Ye^{-\alpha t}, \quad z = Z,$$

where X, Y and Z are the initial position of a fluid parcel and $\alpha > 0$. During the last tutorial, we derived for this flow that it holds, among others,

$$u(X,Y,Z,t) = \alpha X e^{\alpha t}, \quad v(X,Y,Z,t) = -\alpha Y e^{-\alpha t}, \quad w(X,Y,Z,t) = 0,$$
$$u(x,y,z,t) = \alpha x, \quad v(x,y,z,t) = -\alpha y, \quad w(x,y,z,t) = 0.$$

- Let the concentration of a pollutant be defined as $c(x,y,t) = \beta x^2 y e^{-\alpha t}$. Does the concentration change in time for a fluid parcel?
- Evaluate the acceleration in the direction x using both lagrangian and eulerian description.

Solution:
Partial derivative of the concentration with respect to the time would show, if the concentration in a fixed point is changing (it is changing, indeed). To find out if the concentration of a fixed parcel is changing, we fix X, Y and Z and write

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(c(x(X,Y,Z,t), y(X,Y,Z,t), t)\right)\Big|_{X,Y,Z} = \frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} + w\frac{\partial c}{\partial z} =$$
$$-\alpha\beta x^2 y e^{-\alpha t} + (\alpha x)(2\beta x y e^{-\alpha t}) + (-\alpha y)(\beta x^2 e^{-\alpha t}) + 0 = 0$$

The concentration is therefore not changing. The calculation above is actually the computation of the material derivative introduced in the previous lecture.

The fact, that the concentration is not changing with the parcel is however visible also from the lagrangian view: in the description, we can write the concentration as $c(X,Y,Z,t) = \beta X^2 Y$, which is a constant for the parcel.

In the lagrangian description, we can calculate the acceleration by observing, how does the velocity for a fixed particle change:

$$\left(\frac{\partial u(X,Y,Z,t)}{\partial t}\right)_{(X,Y,Z)} = \alpha^2 X e^{\alpha t}.$$

In the eulerian description, the acceleration is defined using the material derivative as

$$\frac{\mathrm{D}u(x,y,z,t)}{\mathrm{D}t} = 0 + u\alpha + 0 + 0 = \alpha u,$$

which is equivalent to the result in lagrangian description. Reason: Material derivative is defined so that it describes lagrangian derivative in a fixed point.

Problem 2.
Decide when the following properties of the material derivatives $\mathrm{D}/\mathrm{D}t$ hold:
a) Formula for the derivative of a sum of two arbitrary smooth functions f, g:

$$\frac{\mathrm{D}}{\mathrm{D}t}(f+g) = \frac{\mathrm{D}}{\mathrm{D}t}f + \frac{\mathrm{D}}{\mathrm{D}t}g.$$

b) Formula for the derivative of product of two arbitrary smooth funcntions f, g:

$$\frac{\mathrm{D}}{\mathrm{D}t}(fg) = f\frac{\mathrm{D}}{\mathrm{D}t}g + g\frac{\mathrm{D}}{\mathrm{D}t}f.$$

c) Exchange of order of the material derivative and the partial spatial derivative of an arbitrary smooth function f:

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\frac{\mathrm{D}f}{\mathrm{D}t}.$$

Solution:
a) The formula for the material derivative is

$$\frac{\mathrm{D}F}{\mathrm{D}t} = \frac{\partial F}{\partial t} + (\mathbf{u}\cdot\nabla)F.$$

The formula for the derivative of the sum therefore obviously holds, thanks to the linearity with respect to F.

b) Regarding the derivative of the product, we must decide, whether

$$(\mathbf{u}\cdot\nabla)(fg) = f(\mathbf{u}\cdot\nabla)g + g(\mathbf{u}\cdot\nabla)f$$

After writing this in components, we see

$$u_i\partial_i(fg) = fu_i\partial_i g + gu_i\partial_i f$$

and the derived formula therefore holds.

c) Left hand side can be written in the form

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial t\partial x} + u\frac{\partial^2 f}{\partial x^2} + v\frac{\partial^2 f}{\partial y\partial x} + w\frac{\partial^2 f}{\partial z\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}\right) -$$
$$-\left(\frac{\partial u}{\partial x}\frac{\partial f}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial f}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial f}{\partial z}\right) = \frac{\partial}{\partial x}\frac{\mathrm{D}f}{\mathrm{D}t} - \frac{\partial \mathbf{u}}{\partial x}\cdot\nabla f.$$

It is therefore impossible to change the order of the derivatives, if the velocity depends on the corresponding spatial coordinate.