4. Inertial oscillation

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Problem 1.
Inertial oscillation is a special type of motion of the air in the atmosphere, in which the inertia of the fluid is balanced by the Coriolis force.
a) This fluid can be described in the eulerian description by equations

$$\frac{\partial u}{\partial t} - fv = 0,$$
$$\frac{\partial v}{\partial t} + fu = 0,$$

where f is the Coriolis parameter, taken as a constant here. Find the period of the oscillations.
b) In the lagrangean description, one can write (outside the equatorial region) equations for motion of an air parcel

$$-K_H|\mathbf{u}|\mathbf{u} \times \mathbf{k} = f\mathbf{u} \times \mathbf{k},$$

where K_H is the horizontal curvature of the motion and $\mathbf{k}$ is a vector pointing upwards in the direction of the z-axis. The equation describes the equality of the centrifugal force created due to the horizontal curvature of the streamlines and the Coriolis force. How does this motion look like in the middle latitudes ($f \approx 10^{-4}$ s^{-1}) with the flow velocity 10 m/s, if the Coriolis parameter is constant? How does the trajectory change, if we consider the dependence of the Coriolis parameter on the latitude $f = 2\Omega \sin(\varphi)$ (Ω is the angular frequency for the rotation of the Earth)?

Solution:
a) In the first step, we will solve the set of differential equations for u and v. One can either guess the solution (insert an einsatz in the equations) or there is a trick:

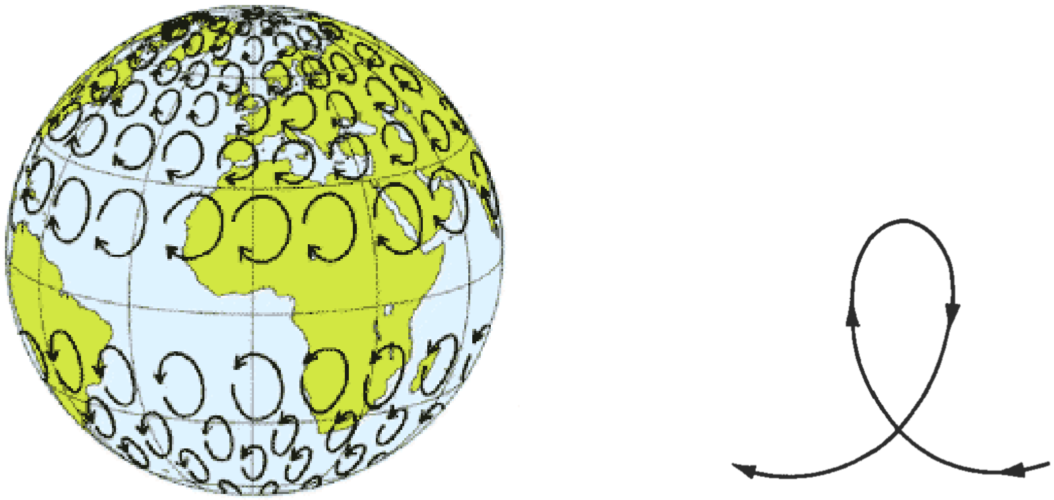
We define $V = u + \mathrm{i}v$, where i is an imaginary constant. The set of equation can be than converted to

$$\frac{\partial V}{\partial t} + \mathrm{i}fV = 0,$$

with the solution $V = V_0 e^{-ift}$. From this, we see that the particles are moving with the period $2\pi/f$. A better insight can be however obtain from the lagrangian description in the part b).

b) If the value of the Coriolis parameter f would not depend on the latitude, the air parcel in the equilibrium state would be moving in circle. The Coriolis vector would point to the centre of the circle and the centrifugal force vector with the same size would head to the opposite direction.

On the northern hemisphere, the Coriolis force is always pointing to the right with respect to the direction of motion. The particle therefore moves in the circle in the clockwise direction. On the southern hemisphere, the Coriolis force is pointing to the left, so the particle moves counterclockwise.



(a) Inertial circulation on the earth.
Taken from https://commons.wikimedia.org/wiki/File:Coriolis_effect14.png#/media/File:Coriolis_effect14.png

(b) The shape of the inertial circculations with the latitudinal changes of the Coriolis parameter. Taken from the book Příručka dynamické meteorologie, Pechala, Bednář

Obrázek 1: Inerční cirkulace.

(On both hemispheres, this is anticyclonal direction - the direction in which motion moves around the pressure high.)

From the equation, we can find other properties of the inertial circulation:

For the size of the velocity vector u_{in}, we have

$$u_{in} = -\frac{f}{K_H} = -fR_{in},$$

where R_{in} is the radius of the inertial circle ($R_{in} < 0$ on the northern hemisphere). For example, if the velocity of the circulation in midlatitudes ($f \approx 10^{-4}$ s^{-1}) is 10 m/s, the radius of the circle is approximately 100 km.

The time to complete one orbit (=inertial period) is given as

$$T_{in} = \frac{2\pi|R_{in|}}{u_{in}} = \frac{2\pi}{|f|}.$$

For the middle latitudes, the resulting inertial period is approximately 17 hours.

Because f grows in the direction from the equatior to poles, for the constant velocity u_{in}, the radius of the oscillation at poles is smaller than near equator. Schematical figure of such oscillations is depicted in Fig. 1a.

In reality, the Coriolis parameter is however varying also during the circulation and the particles are therefore not moving in circles. The curvature is higher further from the equator than at the equator – the resulting trajectory for the northern hemisphere is depicted in Fig. 1b.