5. Fluid dynamics

15. November 2023

Problem 1.
Consider two-dimensional flow described by lagrangian equations

$$x = Xe^{-at}, y = Y + bt,$$

where X and Y specify the original position and a and b are positive constants. Check that the lagrangian and eulerian acceleration coincide.

Solution:
First, we need to get the lagrangian and eulerian velocity. In the lagrangian description, it is

$$\mathbf{u}(X, Y, t) = \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}\right) = \left(-aXe^{-at}, b\right).$$

In the eulerian description, we first need to express $\mathbf{u}(X, Y, t)$ as a function of position x, y, t. Therefore

$$\mathbf{u}(x, y, t) = (-ax, b).$$

The acceleration in lagrangian description can be computed as the derivative of the lagrangian velocity with respect to time:

$$\mathbf{a}(X, Y, t) = \left(a^2 Xe^{-at}, b\right)$$

In the eulerian description, the acceleration is defined as the material derivative of velocity with respect to time, therefore

$$\mathbf{a}(x, y, t) = \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}, \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \left(0 + a^2 x + 0, 0\right),$$

which is the same as in the lagrangian case.

Problem 2.
Stress in Newtonian (Navier-Stokes) equation can be described by the formula

$$\mathbb{T} = -p\mathbb{I} + 2\mu\mathbb{D},$$

where $\mathbb{D}$ is the symmetric part of the velocity gradient defined as $\mathbb{D} = \left(D_{ij}\right)_{i,j=1,2,3}$, $D_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right)$. Show that for the simple shear flow $\mathbf{u} = (u(y), 0, 0)$, all diagonal components of the stress tensor $\mathbb{T}$ are the same. Consider further the Stokes fluid, for which we have (instead of the previous formula)

$$\mathbb{T} = -p\mathbb{I} + \alpha_1\mathbb{D} + \alpha_2\mathbb{D}^2.$$

How does the tensor $\mathbb{T}$ look like this time? Can this have some consequences for the flow? (The equations of motion in this formulation are written as $\rho\, \mathrm{d}\mathbf{u}/\, \mathrm{d}t = \nabla \cdot \mathbb{T} + \rho\mathbf{b}$, where $\mathbf{b}$ is an external force.)

Solution:
For the simple shear flow, we have

$$\mathbb{D} = \frac{1}{2}\begin{pmatrix} 0 & u' & 0 \\ u' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In the newtonian fluid, it therefore holds

$$\mathbb{T} = \begin{pmatrix} -p & \mu u' & 0 \\ \mu u' & -p & 0 \\ 0 & 0 & -p \end{pmatrix},$$

whereas for the Stokes fluid, we get

$$\mathbb{T} = \begin{pmatrix} -p + \frac{\alpha_2}{4}u'^2 & \frac{\alpha_1}{2}u' & 0 \\ \frac{\alpha_1}{2}u' & -p + \frac{\alpha_2}{4}u'^2 & 0 \\ 0 & 0 & -p \end{pmatrix}.$$

The diagonal components in the Stokes fluid are therefore not the same.

From the practical point of view, this is an important difference between the newtonian fluid and some nonnewtonian fluids. There is some stress in the direction perpendicular to the direction of flow. This results in some effects, such as the Weissenberg effect (the fluid is climbing upward on a rotating rod), Barus effect (when the fluid moves out of a narrow space, e.g. the water tap, it widens more than the Newtonian fluid), fluid moving down on an inclined plane creates higher layer at the front, the pressure measured in the hole of a pipe does not correspond to the pressure inside the pipe etc.