7. Circulation

29. November 2023

Problem 1.
Two trucks with the weight $m_T = 5000$ kg and the area of their side $S_T = 20\ \mathrm{m}^2$ are moving next to each other with the velocity $U = 90$ km/h (on the straight road, in windless conditions). Estimate the acceleration given by the difference of the external pressure and the pressure between them. Assume that the distance of the trucks d is similar to their width m and that the air "from the half of their faces" blows to the space between them (Fig. 1) in a vertically homogeneous way.

Solution:
Equivalently to the problem, we can consider standing trucks, with wind moving against them with velocity $-U$. We will denote the external pressure by p_0 and the pressure and the velocity size between the trucks by p_i and U_i.

Based on the continuity equation (conservation of mass), we know that for velocity between the trucks holds $(m + d)U = U_i d$. Therefore,

$$U_i = \left(1 + \frac{m}{d}\right) U.$$

If the flow around the trucks became stationary and we consider the air to be incompressible, we can apply the Bernulli equation in the form

$$\frac{1}{2}\rho U^2 + p_0 = \frac{1}{2}\rho U_i^2 + p_i,$$

where $\rho \approx 1.2\ \mathrm{kg/m^3}$ is the air density. The pressure difference is therefore

$$p_0 - p_i = \frac{1}{2}\rho\left[\left(1 + \frac{m}{d}\right)^2 U^2 - U^2\right] = \rho\frac{m}{d}U^2 + \frac{1}{2}\rho\left(\frac{m}{d}\right)^2 U^2.$$

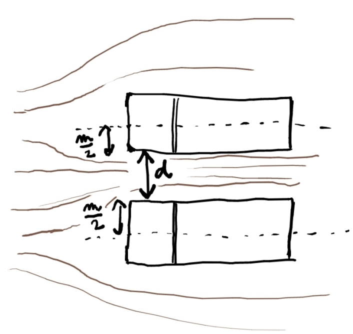

Figure 1: Truck moving next to each other.

This density gradient causes the force $F = (p_0 - p_i)S_T$, acting on each of the trucks. The acceleration of each of the trucks is therefore

$$a = \frac{F}{m_T} = \frac{S_T}{m_T}\left(\rho\frac{m}{d}U^2 + \frac{1}{2}\rho\left(\frac{m}{d}\right)^2 U^2\right).$$

Numerically, this is

$$a \approx \frac{20}{5000}\left(1.2 * 1 * 25^2 + \frac{1.2}{2} * 1^2 * 25^2\right) = 4.5\ \mathrm{m/s^2} = 16.2\ \mathrm{km/h/s}.$$

If the drivers would not adjust the direction of their motion, they would crash relatively fast (given the used approximations).

Problem 2.
Consider the curve $C(t)$ given by the particles of the fluid

$$\mathbf{x} = (a\cos s + a\alpha t\sin s, a\sin s, 0), \quad 0 \le s < 2\pi.$$

By the direct computation, show that

$$\Gamma = \int_{C(t)} \mathbf{u}\cdot \mathrm{d}\mathbf{x} = \int_0^{2\pi} \mathbf{u}\cdot\frac{\partial \mathbf{x}}{\partial s}\,\mathrm{d}s$$

does not depend on time. Why?

Solution:
For the direct computation of the circulation, we first need to find the velocity. This can be computed as the derivative of the vector $\mathbf{x}$ with respect to time with constant initial position of the particle.But taking the initial position constant is equivalent to taking the parameter s constant. Therefore,

$$\mathbf{u} = \left(\frac{\partial \mathbf{x}}{\partial t}\right)_s = (a\alpha\sin s, 0, 0) = (\alpha y, 0, 0).$$

The circulation is then

$$\Gamma = \int_0^{2\pi}\left(-\alpha a^2\sin^2 s + a^2\alpha^2 t\sin s\cos s\right)\mathrm{d}s = -\alpha a^2\pi,$$

which does indeed not depend on time.

Why? Because the curve was defined using the fluid particles, it moves with the fluid, so it is a material curve. We also know that the curve is closed: this follows from $\mathbf{x}(s=0) = (a,0,0) = \mathbf{x}(s=2\pi)$.By the Kelvin theorem, we know that Γ must be constant.