8. Hamiltonian mechanics

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Problem 1.
Consider a Poisson bracket (bilinear mapping from $\mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(M)$, where $\mathcal{F}(M)$ is a space of function on a manifold M) defined by the relation

$$\{F(\mathbf{x}), G(\mathbf{x})\} = \left(\frac{\partial F}{\partial \mathbf{x}}\right)^T L \left(\frac{\partial G}{\partial \mathbf{x}}\right)$$

with so-called Poisson bivector given by the relation

$$L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

a) Show that for $\mathbf{x} = (q, p)$, the Poisson bracket can be written as

$$\{F, G\} = \partial_q F \partial_p G - \partial_p F \partial_q G.$$

b) Show that the equation $\dot{\mathbf{x}} = \{\mathbf{x}, H\}$ with $\mathbf{x} = (q, p)$ is equivalent to the notation of Hamiltonian canonical equations

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}.$$

c) Write the equations of motion for the motion of a free particle in an external field, described by the Hamiltonian $H = p^2/2m + V(q)$.

Solution:
a) From the matrix multiplication

$$\{F, G\} = \begin{pmatrix} \partial_q F & \partial_p F \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial_q G \\ \partial_p G \end{pmatrix} = \begin{pmatrix} \partial_q F & \partial_p F \end{pmatrix} \begin{pmatrix} \partial_p G \\ -\partial_q G \end{pmatrix} = \partial_q F \partial_p G - \partial_p F \partial_q G.$$

b) By expanding the components of the equation $\dot{\mathbf{x}} = \{\mathbf{x}, H\}$:

$$\dot{q} = \{q, H\} = (\partial_q q)\partial_p H - (\partial_p q)\partial_q H = \partial_p H,$$

$$\dot{p} = \{p, H\} = (\partial_q p)\partial_p H - (\partial_p p)\partial_q H = -\partial_q H.$$

c) By substituting of the Hamiltonian:

$$\begin{aligned} \dot{q} =& \frac{p}{m} = v, \\ \dot{p} =& \partial_q V. \end{aligned}$$

These are therefore the standard equations of motion: From the first equation, the velocity is given by the change of the position. The second equation expresses the second Newton law in the form that the change of the momentum is given by the force expressed using the potential V.

Problem 2.
The previous Hamiltonian description can be generalized to an uncanonical form, in which the Poisson bivector is given by a different matrix and we might have a different vector $\mathbf{x}$. An example of such a case can be the rotation of a rigid body that can be described in the spaced formed by the three components of the angular momentum vector $\mathbf{m}$ with the Hamiltonian

$$H(\mathbf{m}) = \frac{1}{2}\left(\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_3^2}{I_3}\right),$$

, where I is the moment of inertia, and a cleverly chosen Poisson bivector $L^{ij} = -m_k \varepsilon_{kij}$, where m_i are components of the angular momentum and ε_{ijk} is the Levi-Civita symbol. This notation is useful as it incomporates certain conservation properties. Using the formalism, decide if the function $|m|^2$ is conserved.

Solution:
In the Hamiltonian mechanics, a quantity is conserved if its Poisson bracket with the Hamiltonian is zero. Starting with the equations of motion in the form $\dot{\mathbf{m}} = \{\mathbf{m}, H\}$, we would get this for a function F from

$$\dot{F} = (\partial_{\mathbf{m}} F) \cdot \dot{\mathbf{m}} = (\partial_{\mathbf{m}} F)^T \dot{\mathbf{m}} = (\partial_{\mathbf{m}} F)^T \{\mathbf{m}, H\} = (\partial_{\mathbf{m}} F)^T (\partial_{\mathbf{m}} \mathbf{m})^T L \partial_{\mathbf{m}} H = (\partial_{\mathbf{m}} F)^T L \partial_{\mathbf{m}} H = \{F, H\}.$$

We therefore need to compute the Poisson bracket $\{|m|^2, H\}$. Using the Poisson bivector, we have

$$\{|\mathbf{m}|^2, H\} = -m_i \varepsilon_{ijk} \left(\partial_{m_j} |\mathbf{m}|^2\right) (\partial_{m_k} H) = -m_i \varepsilon_{ijk} (2m_j) (\partial_{m_k} H) = +m_j \varepsilon_{jik} (2m_i) (\partial_{m_k} H) =$$
$$= +m_j \varepsilon_{jik} \left(\partial_{m_i} |\mathbf{m}|^2\right) (\partial_{m_k} H) = -\{|\mathbf{m}|^2, H\},$$

The quantity is therefore conserved.

It is worth noticing that we did not use the fact that the function H is the Hamiltonian. The relation $\{|\mathbf{m}|^2, E\} = 0$ therefore holds for every function E, which defines a special class of functions called Casimir invariants that play an important role e.g., in the stability theory. The reason is that if you add an arbitrary multiple of the Casimir function to the Hamiltonian, the equations of motion would still hold.

One can further notice that $|m|^2$ is not the only quantity for which this holds: Any function $f(|\mathbf{m}|^2)$ would be a Casimir function, too.

In continuum mechanics, this kind of generalization of the Hamiltonian mechanics is still not enough. To be able to describe a state of fluid, we need to describe properties of all the particles. And the continuum mechanics assumes there are infinitely many fluid particles (at each point of the fluid, there is exactly one particle). This means that the vector $\mathbf{x}$ needs to be infinite-dimensional - it is therefore a function. With this generalisation, the form of equation remains the same. Instead of function, we however have operators, the Poisson bracket works on the space of operators and instead of the derivative, we have the functional derivative.