8. Hamiltonian mechanics

6. December 2023

Problem 1.
Consider a Poisson bracket (bilinear mapping from $\mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(M)$, where $\mathcal{F}(M)$ is a space of function on a manifold M) defined by the relation

$$\{F(\mathbf{x}), G(\mathbf{x})\} = \left(\frac{\partial F}{\partial \mathbf{x}}\right)^T L \left(\frac{\partial G}{\partial \mathbf{x}}\right)$$

with so-called Poisson bivector given by the relation

$$L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

a) Show that for $\mathbf{x} = (q, p)$, the Poisson bracket can be written as

$$\{F, G\} = \partial_q F \partial_p G - \partial_p F \partial_q G.$$

b) Show that the equation $\dot{\mathbf{x}} = \{\mathbf{x}, H\}$ with $\mathbf{x} = (q, p)$ is equivalent to the notation of Hamiltonian canonical equations

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}.$$

c) Write the equations of motion for the motion of a free particle in an external field, described by the Hamiltonian $H = p^2/2m + V(q)$.

Problem 2.
The previous Hamiltonian description can be generalized to an uncanonical form, in which the Poisson bivector is given by a different matrix and we might have a different vector $\mathbf{x}$. An example of such a case can be the rotation of a rigid body that can be described in the spaced formed by the three components of the angular momentum vector $\mathbf{m}$ with the Hamiltonian

$$H(\mathbf{m}) = \frac{1}{2}\left(\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_3^2}{I_3}\right),$$

, where I is the moment of inertia, and a cleverly chosen Poisson bivector $L^{ij} = -m_k \varepsilon_{kij}$, where m_i are components of the angular momentum and ε_{ijk} is the Levi-Civita symbol. This notation is useful as it incomporates certain conservation properties. Using the formalism, decide if the function $|m|^2$ is conserved.