1. Helmholtz decomposition

3. October 2024

Exercise 1.
Consider following velocity vectors:

$$\mathbf{u}_1 = (y, 0, 0), \mathbf{u}_2 = (-y, x, 0), \mathbf{u}_3 = \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 0\right)$$

Plot the velocity fields and compute their curl. In which sense does it hold that in the rotating flow, the velocity curl is non-zero?

Solution:
The fields are depicted in Fig. 1. Velocity $\mathbf{u}_1$ (locally) represents for example the situation of two parallel planar surfaces moving against each other in their planes.

Curl computations:

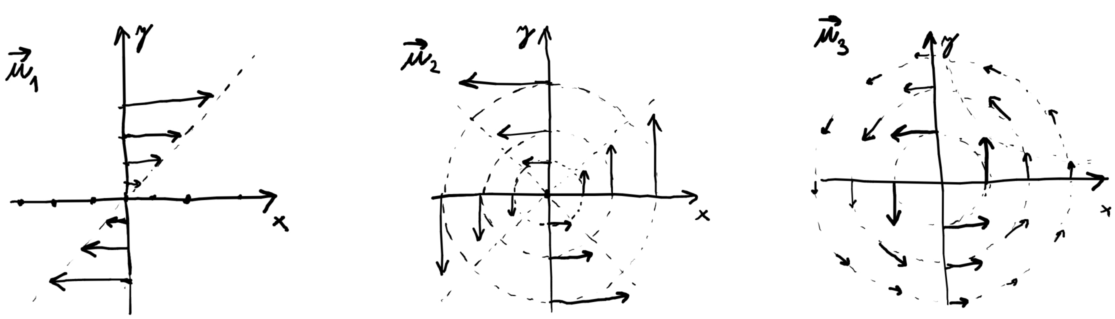
$$\nabla \times \mathbf{u}_1 = (0, 0, -1)$$

$$\nabla \times \mathbf{u}_2 = (0, 0, 2)$$

$$\nabla \times \mathbf{u}_3 = \left(0, 0, \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} + \frac{x^2 + y^2 - 2y^2}{(x^2 + y^2)^2}\right) = (0, 0, 0)$$

Only the field $\mathbf{u}_3$ has therefore zero curl. On the other hand, vector fields $\mathbf{u}_1$ a $\mathbf{u}_2$ are vortical.

Continuum mechanics describes behaviour in individual points. The vector field $\mathbf{u}_1$ is vortical, because it causes "vortical"motion for example around the point $(0, 0, 0)$, similarly as the vector field $\mathbf{u}_2$. On



(a) $\mathbf{u}_1$ (b) $\mathbf{u}_2$ (c) $\mathbf{u}_3$

Obrázek 1: Plots for Exercise 1.

the other hand, vector field $\mathbf{u}_3$ is irrotational, because the particles just "circle"around the point $(0,0,0)$.

Exercise 2.
Consider vector fields from Exercise 1.
a) Are these fields divergent? b) If it holds that the curl of the given vector field is identically zero in $\mathbb{R}^3$, can we deduce something about its divergence? (Think about the Helmholtz decomposition.)

Solution:
a) All the fields from exercise 1 have zero divergence.

b) Yes, in principle. From the Helmholtz decomposition, we know that

$$\mathbf{u} = \nabla\phi + \nabla \times A,$$

where

$$\phi(\mathbf{r}) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\nabla \cdot \mathbf{u}(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}|}\,\mathrm{d}v', \tag{1}$$

$$\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\nabla \times \mathbf{u}(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}|}\,\mathrm{d}v'. \tag{2}$$

This implies that if $\nabla \times \mathbf{u} \equiv 0$, then $\mathbf{A} \equiv \mathbf{0}$, and therefore also $\nabla \times \mathbf{A} \equiv \mathbf{0}$. If we consider nontrivial vector field $\mathbf{u}$, it must hold $\nabla\phi \not\equiv \mathbf{0}$. However, according to Eq. (1), this cannot happen if $\nabla \cdot \mathbf{u} \equiv 0$. Therefore, if it holds for a vector field that both its divergence and curl are zero, the field must be, as a result of the Helmholtz decomposition, identically equal zero.

Why this doesn't hold for a function $\mathbf{u}_3$ from the first exercise: This field is not continuous enough - it is not defined in the origin, Helmholtz decomposition therefore cannot be applied. Helmholtz decomposition in $\mathbb{R}^3$ assumes that the field is at least C^2 (and decays to infinity at least as $1/r$).

Exercise 3.
For the velocity field $\mathbf{u} = (-y, x, 0)$, find an arbitrary scalar and vector potential ϕ and $\mathbf{A}$, so that

$$\mathbf{u} = \nabla\phi + \nabla \times \mathbf{A} \tag{3}$$

inside a bounded domain in $\mathbb{R}^3$.

Solution:
Vector field $\mathbf{u}$ has zero divergence. It holds

$$0 = \nabla \cdot \mathbf{u} = \nabla \cdot \nabla\phi + \nabla \cdot \nabla \times \mathbf{A} = \Delta\phi + 0.$$

Scalar potential ϕ must therefore fulfil the condition $\Delta\phi = 0$. There is a lot of such functions (we are working inside a domain of unspecified shape, so we aren't bothering with any boundary conditions). We can choose for example $\phi = x$. Then $\nabla\phi = (1,0,0)$ and from equation (3) follows

$$\nabla \times \mathbf{A} = \mathbf{u} - \nabla\phi = (-1 - y, x, 0).$$

We are therefore looking for a function $\mathbf{A} = (A^x, A^y, A^z)$ meeting the conditions

$$\begin{pmatrix} \partial_y A^z - \partial_z A^y \\ \partial_z A^x - \partial_x A^z \\ \partial_x A^y - \partial_y A^x \end{pmatrix} = \begin{pmatrix} -1 - y \\ x \\ 0 \end{pmatrix}.$$

Again, there is a lot of such vectors. We can choose for example $A^z = 0$. For the remaining components, we then get $A^x = xz$ a $A^y = (y+1)z$.

Altogether, we've found a scalar potential $\phi = x$ and a vector potential $\mathbf{A} = (xz, (y+1)z, 0)$ that can be used for the decomposition of the vector field $\mathbf{u}$.