2. Air properties

17. October 2024

Exercise 1.
A tourist is walking in the mountains with a barometer, measuring the pressure. In the valley, he got 1050 hPa, on the mountain, he got 950 hPa.

- Using the baric step near the ocean surface (i.e., with the highest air density), what is the estimated height of the mountain?
- The tourist had also a thermometer and he was measuring the average temperature during the hike, which was 0°C. Using the Babinet formula, what would be the height difference. Refine the solution using the Laplace formula.
- If the tourist measured the same pressures in summer with the average temperature 20°C, how high would the mountain be?

Solution:
The baric step near the ocean surface is approximately 8 m on 1 hPa. This would make the height of 800 m for the measured difference of 100 hPa.

The temperature 0°C is 273.15 K. The Babinet formula is

$$z_2 - z_1 = 16000 \left(1 + 0.00366\overline{T}\right) \frac{p_1 - p_2}{p_1 + p_2}, \tag{1}$$

which gives the difference approximately 1600 m. The fact that it is greater than the previous estimate is reasonable, because we used the baric step corresponding to the lowest level only, although the baric step increases with height.

However, with this height difference, the Babinet formula might get imprecise. The Laplace formula for the height difference is

$$z_2 - z_1 = 8000 \left(1 + 0.00366\overline{T}\right) \ln \frac{p_1}{p_2}. \tag{2}$$

Substituting into it gives the height difference 1601 m, which is not much different to the previous result.

If the temperature was 20°C higher, the Laplace formula would give the mountain height 1660 m.

Exercise 2.
An eagle wants to eat a mouse. Before diving to the mouse, the eagle cries. Can the mouse be warned by the sound? Estimate, what time does the sound take until it reaches the mouse.

Near the ground, the temperature is $T_1 = 40$°C, the eagle flies in the height 50 m above the flat surface, where the temperature is $T_2 = 20$°C. Assume for simplicity, that the temperature changes between T_1 and T_2 at an interface at the height $z_1 = 10$ m. That is, there are two layers of air: the first layer with temperature T_1 starts at ground and goes to the height z_1, the second layer with temperature T_2 continues to the height of the eagle with the thickness $z_2 = 40$ m. The horizontal distance of the eagle to the mouse is $m = 5$ m.

Think further about the situation with linear vertical temperature profile instead of just two values. Would this improve the chances for the mouse?

Solution:
If the sound travels to the eagle on the shortest way, it travels the distance

$$\sqrt{z_2^2 + \left(\frac{z_2}{z_1 + z_2} m\right)^2}$$

in the top layer and the distance

$$\sqrt{z_1^2 + \left(\frac{z_1}{z_1 + z_2} m\right)^2}$$

in the lower one.

Because the speed of sound depends on the temperature T by the formula

$$c = \sqrt{\kappa r T},$$

where $\kappa = 1.4$ and the specific gas constant of air is $r = 287$ J/kg/K, it is different in both layers. In particular, for the top layer, it is $c_2 = \sqrt{1.4 \times 287 \times 293} = 343$ m/s and $c_1 = \sqrt{1.4 \times 287 \times 313} = 355$ m/s in the lower level.

The sound would therefore travel for

$$t_0 = \frac{\sqrt{z_1^2 + \left(\frac{z_1}{z_1+z_2} m\right)^2}}{\sqrt{\kappa r T_1}} + \frac{\sqrt{z_2^2 + \left(\frac{z_2}{z_1+z_2} m\right)^2}}{\sqrt{\kappa r T_2}} \approx 0.02834 + 0.11716 \approx 0.14550\,\mathrm{s}.$$

However, this is not the shortest time in which the sound arrives to the mouse. Because the sound is faster in the air with higher temperature, it is more efficient for it to travel greater distance in the lower level and smaller distance in the upper layer. We therefore need to optimise for the point on the interface, so that the time is minimal (x in Fig. 1).

In this general case, time before the sound reaches the eagle is

$$t(x) = \frac{\sqrt{z_1^2 + (m-x)^2}}{\sqrt{\kappa r T_1}} + \frac{\sqrt{z_2^2 + x^2}}{\sqrt{\kappa r T_2}}. \tag{3}$$

To get a minimal time, we need to find x for which the derivative of the previous expression equals 0. That leads to equality

$$0 = \frac{1}{\sqrt{\kappa r T_1}} \frac{m-x}{\sqrt{z_1^2 + (m-x)^2}} - \frac{1}{\sqrt{\kappa r T_2}} \frac{x}{\sqrt{z_2^2 + x^2}}.$$

This however hard to solve analytically for x, which is what we wanted. However, based on the Figure 1, we can see that the fractions in both summands are actually sinuses of angles plotted in the figure.

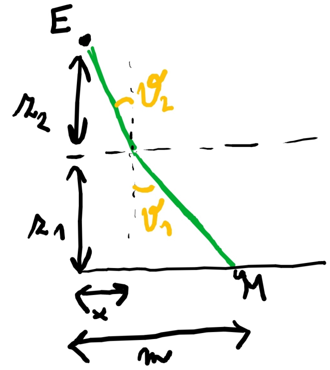

Figure 1: Scheme of the problem.

We thus got a law for the sound refraction

$$\frac{\sin\theta_1}{\sqrt{T_1}} = \frac{\sin\theta_2}{\sqrt{T_2}}.$$

This is a law very similar to the refraction of light in optics, with the acoustic index of refraction $n = 1/\sqrt{T}$.

If T_1 is greater than T_2 (our situation), implies that θ_1 is greater than θ_2. That is, if the sound goes from an environment with lower temperature to the environment with higher temperature, the angle increases.

Returning back to our problem, we know know the trajectory of the sound. Or at least, we know how it is refracted - we could not analytically solve the equations for x! This creates a problem for the calculation of the path. But luckily, the angle at which the eagle looks at the mouse is very small, allowing for brutal approximation $\sin\theta \approx \theta$ and $\tan\theta \approx \theta$. From that, we can simplify geometric relationships $x = z_2 \tan\theta_2$ and $x = m - z_1 \tan\theta_1$ and the refraction law to the system of equations:

$$\theta_2 = \frac{x}{z_2}, \quad \theta_1 = \frac{m-x}{z_1}, \quad \theta_1 = \sqrt{\frac{T_1}{T_2}}\theta_2.$$

This system can be easily solved for x:

$$x = m\frac{z_2}{z_2 + z_1\sqrt{\frac{T_1}{T_2}}} \approx 3.97,$$

which slightly differs from 4 without the temperature modification. Finally, this can be substituted to the equation (3) to give $t_{min} = 0.02834 + 0.11715 = 0.14549$ s. This is by 0.01 ms shorter than our previous estimate, so the mouse would have more time to flee.

In our case, the situation was however simple with just two temperature levels. With a continuous distribution, the refraction can lead to the situation that the sound would not be heard at certain places - a shadow would be created, see Fig. 2. If the mouse sits in the shadow, it would not get any warning about the eagle.

For comparison, the refractive index of electromagnetic rays depends on the density of air, decreasig with its decrease. In the usual situation in the atmosphere, when both density and temperature decrease with height, the rays are slanted in a different way. You will learn more about it at the subject of our department Propagation of Acoustic and Electromagnetic Waves in Atmosphere.

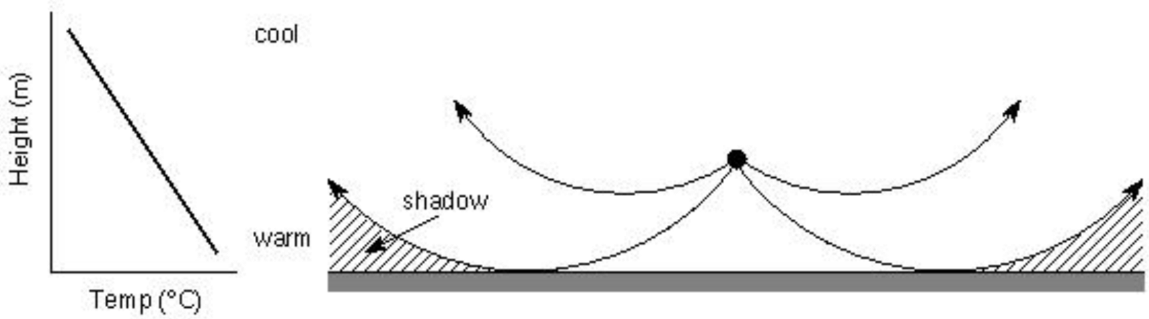

Figure 2: Sound propagation with continuous profile.