3. Trajectory and streamline

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Problem 1.
Consider flow that is described in a lagrangian way using equations

$$x = Xe^{\alpha t}, \quad y = Ye^{-\alpha t}, \quad z = Z,$$

where X, Y and Z are the initial position of a fluid parcel and $\alpha > 0$.

- Find its trajectory.
- Find and plot its streamlines.
- Decide whether the flow is stationary.
- Let the concentration of a pollutant be defined as $c(x, y, t) = \beta x^2 y e^{-\alpha t}$. Does the concentration change in time for a fluid parcel?
- Evaluate the acceleration in the direction x using both lagrangian and eulerian description.

Solution:
To find the trajectory, it is necessary to eliminate time in the lagrangian description of the flow. By substituting the factor $e^{\alpha t}$ from function x to y, we get:

$$x = \frac{XY}{y}, \quad z = Z.$$

Trajectories are therefore hyperbolas lying in the xy plane.

Streamlines are defined as curves, whose tangent line is (in every point) the velocity vector. To find the streamlines, we therefore need to find how the velocity looks like. In lagrangian description, the velocity is defined as

$$\begin{aligned}
u(X, Y, Z, t) &= \left(\frac{\partial x}{\partial t}\right)_{(X,Y,Z)} = \alpha X e^{\alpha t}, \\
v(X, Y, Z, t) &= \left(\frac{\partial y}{\partial t}\right)_{(X,Y,Z)} = -\alpha Y e^{-\alpha t}, \\
w(X, Y, Z, t) &= \left(\frac{\partial z}{\partial t}\right)_{(X,Y,Z)} = 0.
\end{aligned}$$

To switch from the description to eulerian, we have to express individual components as functions of x, y and z. Therefore,

$$u(x, y, z, t) = \alpha x, \quad v(x, y, z, t) = -\alpha y, \quad w(x, y, z, t) = 0.$$

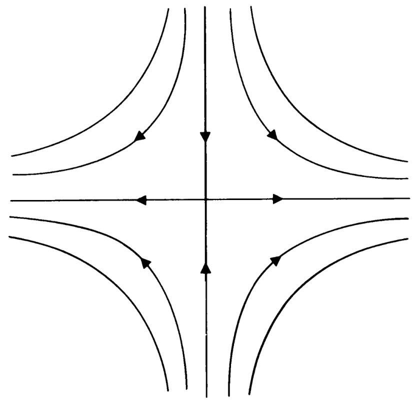

Figure 1: Streamlines for exercise 1.

Plotting these velocities in (infinite number of) different points, we get the streamlines, see Fig. 1.

These streamlines are again hyperbolas, as well as the trajectories. This is not a coincidence – because the eulerian velocities are not dependent on time, the flow is stationary and the trajectories thus coincide with streamlines.

Partial derivative of concentration with respect to time would find, if the concentration in a fixed point is changing (it is changing, indeed). To find out if the concentration of a fixed parcel is changing, we fix X, Y and Z and write

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(c(x(X,Y,Z,t),y(X,Y,Z,t),t)\right)\Big|_{X,Y,Z} = \frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} + w\frac{\partial c}{\partial z} = \\ -\alpha\beta x^2 y e^{-\alpha t} + (\alpha x)(2\beta x y e^{-\alpha t}) + (-\alpha y)(\beta x^2 e^{-\alpha t}) + 0 = 0$$

The concentration is therefore not changing. The calculation above is actually the computation of the material derivative.

The fact, that the concentration is not changing with the parcel is however visible also from the lagrangian view: in the description, we can write the concentration as $c(X,Y,Z,t) = \beta X^2 Y$, which is a constant for the parcel.

In the Lagrangian view, the acceleration can be computed based on the information, how the velocity changes for a fixed particel:

$$\left(\frac{\partial u(X,Y,Z,t)}{\partial t}\right)_{(X,Y,Z)} = \alpha^2 X e^{\alpha t}.$$

In the Eulerian description, the acceleration is defined using the material derivative as

$$\frac{\mathrm{D}u(x,y,z,t)}{\mathrm{D}t} = 0 + u\alpha + 0 + 0 = \alpha u,$$

which is equivalent to the lagrangian results. Reason: The material derivative is defined such that it describes the lagrangian derivative in a fixed point.

Problem 2.
Consider the nonstationary flow

$$u = u_0, \quad v = kt, \quad , w = 0,$$

where u_0 and k are positive constants. Derive how the streamlines look like and find the trajectories.

Solution:
Streamlines $x = x(s)$, $y = y(s)$ a $z = z(s)$ are in every time t described by equations

$$\frac{\frac{\mathrm{d}x(s)}{\mathrm{d}s}}{u} = \frac{\frac{\mathrm{d}y(s)}{\mathrm{d}s}}{v} = \frac{\frac{\mathrm{d}z(s)}{\mathrm{d}s}}{w}$$

(their direction should be the same as the ratio of the velocity components). We thus want to compute

$$\frac{\frac{\mathrm{d}y(s)}{\mathrm{d}s}}{\frac{\mathrm{d}x(s)}{\mathrm{d}s}} = \frac{v}{u}.$$

Using the chain rule, the left-hand side is $\mathrm{d}y/\,\mathrm{d}x$, whereas the right-hand side is, after substituting the velocity components, kt/u_0. By integration of the remaining differential equations, we get:

$$y = \frac{kt}{u_0}x + const.$$

The second equality in the definition implies:

$$z = const.$$

Streamlines are therefore at every time instant straight lines in the xy plane.

As for the trajectories, it is possible to obtain them from integrals of velocity in lagrangian description (constant initial positions of the parcels) with respect to time:

$$\left(\frac{\partial x}{\partial t}\right)_{\mathbf{X}} = u_0, \quad \left(\frac{\partial y}{\partial t}\right)_{\mathbf{X}} = kt, \quad \left(\frac{\partial z}{\partial t}\right)_{\mathbf{X}} = 0,$$

therefore

$$x = u_0 t + F_1(\mathbf{X}), \quad y = \frac{1}{2}kt^2 + F_2(\mathbf{X}), \quad z = F_3(\mathbf{X}).$$

Functions F_1, F_2 and F_3 are specified so that at time $t = 0$, it is $\mathbf{x} = \mathbf{X}$. The motion of the parcels can be therefore described by equations

$$x = u_0 t + X, \quad y = \frac{1}{2}kt^2 + Y, \quad z = Z.$$

Finally, we get the trajectory from the elimination of time:

$$y = \frac{1}{2}k\left(\frac{X - x}{u_0}\right)^2 + Y$$

Although the streamlines at a fixed time are straight-lines, the parcels are moving along parabolas.

Problem 3.
Decide when the following properties of the material derivatives $\mathrm{D}/\,\mathrm{D}t$ hold:
a) Formula for the derivative of a sum of two arbitrary smooth functions f, g:

$$\frac{\mathrm{D}}{\mathrm{D}t}(f + g) = \frac{\mathrm{D}}{\mathrm{D}t}f + \frac{\mathrm{D}}{\mathrm{D}t}g.$$

b) Formula for the derivative of product of two arbitrary smooth funcntions f, g:

$$\frac{\mathrm{D}}{\mathrm{D}t}(fg) = f\frac{\mathrm{D}}{\mathrm{D}t}g + g\frac{\mathrm{D}}{\mathrm{D}t}f.$$

c) Exchange of order of the material derivative and the partial spatial derivative of an arbitrary smooth function f:

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\frac{\mathrm{D}f}{\mathrm{D}t}.$$

Solution:
a) The formula for the material derivative is

$$\frac{\mathrm{D}F}{\mathrm{D}t} = \frac{\partial F}{\partial t} + (\mathbf{u}\cdot\nabla)F.$$

The formula for the derivative of the sum therefore obviously holds, thanks to the linearity with respect to F.

b) Regarding the derivative of the product, we must decide, whether

$$(\mathbf{u}\cdot\nabla)(fg) = f(\mathbf{u}\cdot\nabla)g + g(\mathbf{u}\cdot\nabla)f$$

After writing this in components, we see

$$u_i\partial_i(fg) = fu_i\partial_i g + gu_i\partial_i f$$

and the derived formula therefore holds.

c) Left hand side can be written in the form

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial t\partial x} + u\frac{\partial^2 f}{\partial x^2} + v\frac{\partial^2 f}{\partial y\partial x} + w\frac{\partial^2 f}{\partial z\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}\right) -$$
$$- \left(\frac{\partial u}{\partial x}\frac{\partial f}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial f}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial f}{\partial z}\right) = \frac{\partial}{\partial x}\frac{\mathrm{D}f}{\mathrm{D}t} - \frac{\partial \mathbf{u}}{\partial x}\cdot\nabla f.$$

It is therefore impossible to change the order of the derivatives, if the velocity depends on the corresponding spatial coordinate.