4. Incompressibility

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Problem 1.
Consider two-dimensional flow described by lagrangian equations

$$x = Xe^{-at}, y = Y + bt,$$

where X and Y specify the original position and a and b are positive constants. Check that the lagrangian and eulerian acceleration coincide.

Solution:
First, we need to get the lagrangian and eulerian velocity. In the lagrangian description, it is

$$\mathbf{u}(X, Y, t) = \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}\right) = \left(-aXe^{-at}, b\right).$$

In the eulerian description, we first need to express $\mathbf{u}(X, Y, t)$ as a function of position x, y, t. Therefore

$$\mathbf{u}(x, y, t) = (-ax, b).$$

The acceleration in lagrangian description can be computed as the derivative of the lagrangian velocity with respect to time:

$$\mathbf{a}(X, Y, t) = \left(a^2 Xe^{-at}, b\right)$$

In the eulerian description, the acceleration is defined as the material derivative of velocity with respect to time, therefore

$$\mathbf{a}(x, y, t) = \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}, \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \left(0 + a^2 x + 0, 0\right),$$

which is the same as in the lagrangian case.

Problem 2.
Let us have a general velocity field $\mathbf{u}$ with the density ρ. Consider a volume $\mathcal{V}(t)$ inside the fluid in time t composed of particles that take the volume $V(t_0)$ at the initial time t_0. How can be expressed the fact that the field is incompressible? What does it mean for the velocity? What does it mean for the density?

Solution:
If the fluid cannot be expressed, we should have $\mathcal{V}(t) = V(t_0)$ for every time t. Therefore, it must be also

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}(t)} 1 \,\mathrm{d}v = 0.$$

We would like to move the differentiation inside the integral. We however need to rewrite first the integral so that the domain over which we are integrating does not depend on t. We therefore consider mapping $\mathcal{X}$ that maps initial positions $\mathbf{X}$ of particles to their positions $\mathbf{x}$ in time t. That is, $\mathbf{x} = \mathcal{X}(\mathbf{X}, t)$ and $\mathcal{V}(t) = \mathcal{X}(V(t_0), t)$.

We will denote the Jacobian tensor $\partial\mathcal{X}(\mathbf{X}, t)/\partial\mathbf{X}$ of the transformation by $\mathbb{F}$ (=deformation gradient). This tensor describes the change of shape – for example from the Taylor expansion, we have for a 1D deformation

$$\mathrm{d}\mathbf{x} = \mathbf{x}_2 - \mathbf{x}_1 = \mathcal{X}(\mathbf{X}_1 + \mathrm{d}\mathbf{X}, t) - \mathcal{X}(\mathbf{X}_1, t) = \frac{\partial\mathcal{X}}{\partial\mathbf{X}}(\mathbf{X}_1, t)\,\mathrm{d}\mathbf{X} + \mathcal{O}((\mathrm{d}\mathbf{X})^2).$$

For 3D, this results in the formula $\mathrm{d}v = (\det\mathcal{F})\,\mathrm{d}V$ (this follows from the property of determinant expressing the change of the volume after modifying it by multiplication with the matrix).

In any case, we rewrite the first integral as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}(t)} 1\,\mathrm{d}v = \frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t_0)} \det\mathbb{F}\,\mathrm{d}V = \int_{V(t_0)} \frac{\mathrm{d}}{\mathrm{d}t}(\det\mathbb{F})\,\mathrm{d}V$$

From the mathematical point of view, it can be derived that, in the sense of Gateaux derivative, $(\partial(\det)\mathbb{A}/\partial\mathbb{A})[\mathbb{B}] = (\det\mathbb{A})\mathrm{Tr}(\mathbb{A}^{-1}\mathbb{B})$, which would finally lead to the equality $\mathrm{d}(\det\mathbb{F})/\,\mathrm{d}t = (\det\mathbb{F})\nabla\cdot\mathbf{u}$, where $\mathbf{u}$ is the velocity. The determinant of $\mathbb{F}$ can be than used to convert the integral back to the volume $\mathcal{V}(t)$. Therefore, the fluid is incompressible if $\nabla\cdot\mathbf{u} = 0$ (this is true even without the integral because it holds for any volume $\mathcal{V}(t)$).

However, we can see that also using some physical consideration of the volume changes. If we look at a volume, it is clear that the volume increases if the particles at the surface move away from each other and not if they, for example, move in one direction. Therefore, the changes can be described by integrating the velocity over the surface of the volume. And due to the Gauss theorem, this is equivalent to the divergence of the velocity:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}(t)} 1\,\mathrm{d}v = \int_{\mathcal{S}(t)} \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}s = \int_{\mathcal{V}(t)} \nabla\cdot\mathbf{u}\,\mathrm{d}v,$$

where $\mathbf{n}$ is the normal to the volume surface $\mathcal{S}(t)$. The last equality can also be seen without the knowledge of the Gauss theorem: The divergence would give a value for each point of the volume. But the values corresponding to the changes inside the volume would cancel out, leaving only the contribution from the velocity component normal to the boundary. And because the volume integral "contains one more integration"compared to the surface one, this cancels out the divergence.

All in all, we saw that the incompressibility means that the divergence of the velocity is zero. Let us look what does this mean in the context of the continuity equation. We will first rederive the equation, using the formulas derived above. The continuity equation is based on the fact that the mass of given particles does not change. Therefore, we have

$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}(t)} \rho\,\mathrm{d}v = \frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t_0)} \rho\det\mathbb{F}\,\mathrm{d}V &= \int_{V(t_0)} \rho\,\frac{\mathrm{d}}{\mathrm{d}t}(\det\mathbb{F}) + \frac{\mathrm{d}\rho}{\mathrm{d}t}\det\mathbb{F}\,\mathrm{d}V \\ = \int_{V(t_0)} \rho\,(\nabla\cdot\mathbf{u})\det\mathbb{F} + \frac{\mathrm{d}\rho}{\mathrm{d}t}\det\mathbb{F}\,\mathrm{d}V &= \int_{\mathcal{V}(t)} \rho\,(\nabla\cdot\mathbf{u}) + \frac{\mathrm{d}\rho}{\mathrm{d}t}\,\mathrm{d}v \end{aligned}$$

The continuity equation is therefore

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho\nabla\cdot\mathbf{u},$$

as derived already at the lecture. From this, it follows that the incompressible flow also satisfied $\mathrm{d}\rho/\mathrm{d}t \equiv 0$ – the material derivative of density does not change. This is often taken as an equivalent characterisation of the incompressibility. Nevertheless, if the system does not conserve mass, these two concepts are not equivalent.

Problem 3.
Consider the flow with velocity $\mathbf{u} = (x, y, 0)$. Is this flow incompressible?

Solution:
No, this flow is not incompressible, as its divergence is non-zero.