5. Fluid dynamics

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Problem 1.
Stress in Newtonian (Navier-Stokes) equation can be described by the formula

$$\mathbb{T} = -p\mathbb{I} + 2\mu\mathbb{D},$$

where $\mathbb{D}$ is the symmetric part of the velocity gradient defined as $\mathbb{D} = (D_{ij})_{i,j=1,2,3}$, $D_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$. Show that for the simple shear flow $\mathbf{u} = (u(y), 0, 0)$, all diagonal components of the stress tensor $\mathbb{T}$ are the same. Consider further the Stokes fluid, for which we have (instead of the previous formula)

$$\mathbb{T} = -p\mathbb{I} + \alpha_1\mathbb{D} + \alpha_2\mathbb{D}^2.$$

How does the tensor $\mathbb{T}$ look like this time? Can this have some consequences for the flow? (The equations of motion in this formulation are written as $\rho\, \mathrm{d}\mathbf{u}/\,\mathrm{d}t = \nabla \cdot \mathbb{T} + \rho\mathbf{b}$, where $\mathbf{b}$ is an external force.)

Solution:
For the simple shear flow, we have

$$\mathbb{D} = \frac{1}{2}\begin{pmatrix} 0 & u' & 0 \\ u' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In the newtonian fluid, it therefore holds

$$\mathbb{T} = \begin{pmatrix} -p & \mu u' & 0 \\ \mu u' & -p & 0 \\ 0 & 0 & -p \end{pmatrix},$$

whereas for the Stokes fluid, we get

$$\mathbb{T} = \begin{pmatrix} -p + \frac{\alpha_2}{4}u'^2 & \frac{\alpha_1}{2}u' & 0 \\ \frac{\alpha_1}{2}u' & -p + \frac{\alpha_2}{4}u'^2 & 0 \\ 0 & 0 & -p \end{pmatrix}.$$

The diagonal components in the Stokes fluid are therefore not the same.

From the practical point of view, this is an important difference between the newtonian fluid and some nonnewtonian fluids. There is some stress in the direction perpendicular to the direction of flow. This results in some effects, such as the Weissenberg effect (the fluid is climbing upward on a rotating rod), Barus effect (when the fluid moves out of a narrow space, e.g. the water tap, it widens more than the Newtonian fluid), fluid moving down on an inclined plane creates higher layer at the front, the pressure measured in the hole of a pipe does not correspond to the pressure inside the pipe etc.

Problem 2.
Decide, if the parity described by the following equations

$$\mathbf{r}' = -\mathbf{r},$$
$$t' = t,$$
$$p' = p,$$

is a symmetry for the Euler equations containing the gravity

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p - g\mathbf{k},$$

where $\mathbf{k}$ is a unit vector in the direction z. If not, how do we have to modify the transformation, so that it is a symmetry?

Solution:
With the transformation, it obviously holds (by the chain rule)

$$\partial_t' = \partial_t,$$
$$\mathbf{u}' = -\mathbf{u},$$
$$\nabla' = -\nabla.$$

Equation

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla p - g\mathbf{k}$$

can be therefore rewritten as

$$-\partial_t' \mathbf{u}' - \mathbf{u}' \cdot \nabla' \mathbf{u}' = +\frac{1}{\rho}\nabla' p' - g\mathbf{k},$$

and therefore

$$\partial_t' \mathbf{u}' + \mathbf{u}' \cdot \nabla' \mathbf{u}' = -\frac{1}{\rho}\nabla' p' + g\mathbf{k}.$$

The equation with the dashes differs therefore from the equation without dashes in the sign of the term $g\mathbf{k}$. Given transform is thus not a symmetry of the system - it is not a symmetry for the vertical component of the Euler equations.

To get the symmetry and conserve the character of the transform (the simplest transform would be of course to change it to identity), we can modify the equation of the pressure force. If we write $p' = p + \tilde{p}$ instead of the original formula $p' = p$, we would get the term $+\nabla'\tilde{p}/\rho$ in the last equation. The function $\tilde{p}$ would therefore satisfy

$$\frac{1}{\rho}\nabla'\tilde{p} + g\mathbf{k} = -g\mathbf{k}.$$

With the transform of the gradient to the coordinates without dashes, this is equivalent with the formula

$$\frac{1}{\rho}\nabla\tilde{p} = 2g\mathbf{k}.$$

By integration, we can get for example (with zero integration constants) $\tilde{p} = 2\rho g z$.

To conserve the symmetry, we therefore need to transform the pressure by the formula $p' = p + 2\rho g z$. This makes sense: for the hydrostatic equilibrium, if the density is constant, we would have equilibrium pressure $\rho g z$. If we changed the direction of the z coordinate, this value change from plus to minus, in total twice.