6. Circulation

14. November 2024

Problem 1.
Consider the curve $C(t)$ given by the particles of the fluid

$$\mathbf{x} = (a\cos s + a\alpha t \sin s, a \sin s, 0), \quad 0 \le s < 2\pi.$$

By the direct computation, show that

$$\Gamma = \int_{C(t)} \mathbf{u} \cdot \mathrm{d}\mathbf{x} = \int_0^{2\pi} \mathbf{u} \cdot \frac{\partial \mathbf{x}}{\partial s} \, \mathrm{d}s$$

does not depend on time. Why?

Solution:
For the direct computation of the circulation, we first need to find the velocity. This can be computed as the derivative of the vector $\mathbf{x}$ with respect to time with constant initial position of the particle.But taking the initial position constant is equivalent to taking the parameter s constant. Therefore,

$$\mathbf{u} = \left(\frac{\partial \mathbf{x}}{\partial t}\right)_s = (a\alpha \sin s, 0, 0) = (\alpha y, 0, 0).$$

The circulation is then

$$\Gamma = \int_0^{2\pi} \left(-\alpha a^2 \sin^2 s + a^2\alpha^2 t \sin s \cos s\right) \mathrm{d}s = -\alpha a^2 \pi,$$

which does indeed not depend on time.

Why? Because the curve was defined using the fluid particles, it moves with the fluid, so it is a material curve. We also know that the curve is closed: this follows from $\mathbf{x}(s = 0) = (a, 0, 0) = \mathbf{x}(s = 2\pi)$.By the Kelvin theorem, we know that Γ must be constant.

Problem 2.
Compute the vorticity for the flow in the previous problem. Integrate the vorticity for time $t = 0$ over the area defined by the curve $C(t)$. Explain the result. How will the area enclosed by the curve $C(t)$ evolve in time $t > 0$?

Solution:
Vorticity is defined as $\partial v/\partial x - \partial u/\partial y$. It is therefore

$$\omega = -\alpha.$$

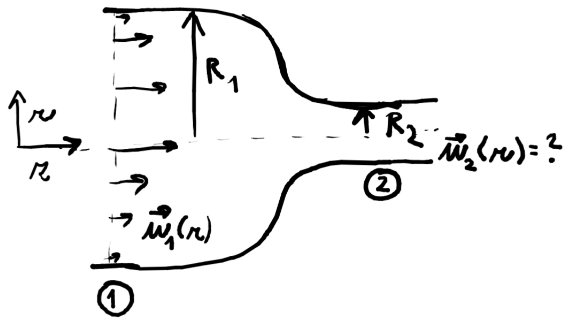

Figure 1: Scheme for the Problem 3.

In time $t = 0$, the curve is a circle with radius a. It holds

$$\int_{\mathrm{Int}(C(0))} \omega \,\mathrm{d}\mathbf{x} = -\alpha |\mathrm{Int}(C(0))| = -\alpha \pi a^2,$$

which is the same as the already calculated circulation: the Stokes theorem implies

$$\int_C \mathbf{u} \cdot \mathrm{d}\mathbf{s} = \int_{\mathrm{Int}(C)} (\nabla \times \mathbf{u}) \cdot d\mathbf{A},$$

For the 2D flow with the curl nonzero only in the direction z, this equals the integral of the vorticity over the area encompassed by the curve $C(t)$. For the flow on a plane, it is therefore possible to express the circulation as

$$\Gamma = \int_A \omega \,\mathrm{d}A.$$

Regarding the evolution for $t > 0$, we know that the vorticity does not depend on time and the area can be therefore written as $A(t) = \Gamma/(-\alpha)$. Because we know by the Kelvin circulation theorem that the circulation is constant, the area will also be constant and equal to πa^2.

Problem 3.
Consider an inviscid, incompressible flow in a cylindrical pipe depicted on Fig. 1. Velocity in the zone 1 is described as $\mathbf{u}_1 = (0, 0, u_1)$ with

$$u_1(r) = U_1 \left(1 - \left(\frac{r}{R_1} \right)^2 \right). \tag{1}$$

Compute the vorticity and the velocity in the zone 2.

Solution:
If the velocity was constant, the solution would be to simply use the continuity equation, so that the fluid from the thick pipe fit into the narrow pipe. However, the flow is more complicated in our case.

Instead, we will use the Kelvin circulation theorem. Let's start with calculating the vorticity in the zone 1. In our settings, it is useful to use the cylindrical coordinates. This means that the vorticity $\omega = \nabla \times \mathbf{u}$ needs to be written down using the operator ∇ for the cylindrical coordinates (r, θ, z):

$$\nabla \equiv \left(\frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \theta}, \frac{\partial}{\partial z} \right). \tag{2}$$

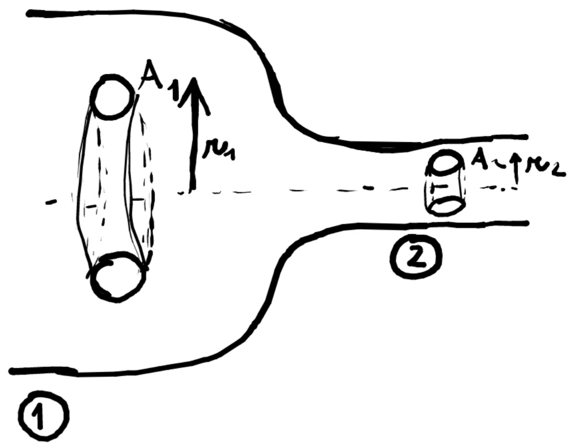

Figure 2: Definition of the torus.

Considering the symmetry of the situation ($\partial/\partial\theta = 0$ and $u_\theta = u_r = 0$), we have for the velocity $\mathbf{u} = (u_r, u_\theta, u_z) = (0, 0, u_z)$ formula for the vorticity

$$\omega = \left(0, -\frac{\partial u_z}{\partial r}, 0\right). \tag{3}$$

In the zone 1, we thus have the nonzero component in the θ-direction $\omega = (0, \omega_1, 0) = \left(0, -\frac{\partial u_1}{\partial r}, 0\right)$. Using the prescription for u_1, we get

$$\omega_1 = -\frac{\partial u_1}{\partial r} = U_1 \frac{2r}{R_1^2}. \tag{4}$$

Lines tangential to the vorticity, so-called vortex lines, analogy of the streamlines for velocity, are therefore circles around the center of the pipe. Using this information, we can define vortex rings, torus in the zone 1 with the boundary formed by the vortex lines as defined in Fig. 2. As a corollary of the Kelvin circulation theorem, vortex lines move with the fluid, there are material curves. For the ring, this can be proven since any closed curve on the ring surface would have zero circulation, as the vorticity is tangential to it. Kelvin circulation theorem implies that the circulation remains zero if the ring moves with the fluid. This means that the curve remains on the ring as no other vortex line can pass through it.

If we follow the same tube, the Kelvin circulation theorem

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} \int_A \omega \, \mathrm{d}A \tag{5}$$

implies

$$\omega_1 A_1 = \omega_2 A_2 \tag{6}$$

for ω_2 being analogically the vorticity in the direction θ in the zone 2. We can also consider mass conservation for the ring (volume of the ring remains constant):

$$A_1 2\pi r_1 = A_2 2\pi r_2. \tag{7}$$

This implies the formula for the vorticity in the second zone

$$\omega_2 = \frac{r_2}{r_1}\omega_1 = \frac{r}{r_1} U_1 \frac{2r_1}{R_1^2} = U_1 \frac{2r}{R_1^2}. \tag{8}$$

By inverting the equation

$$U_1 \frac{2r}{R_1^2} = \omega_2 = -\frac{\partial u_2}{\partial r}, \tag{9}$$

we can get the velocity

$$u_2 = -U_1 \frac{r^2}{R_1^2} + c. \tag{10}$$

Finally, we have to evaluate the constant c. To this end, we can use the conservation of mass in the whole sections 1 and 2:

$$\int_0^{R_1} u_1 2\pi r \, \mathrm{d}r = \int_0^{R_2} u_2 2\pi r \, \mathrm{d}r, \tag{11}$$

implying

$$\int_0^{R_1} U_1 \left(1 - \left(\frac{r}{R_1}\right)^2\right) 2\pi r \, \mathrm{d}r = \int_0^{R_2} \left(-U_1 \frac{r^2}{R_1^2} + c\right) 2\pi r \, \mathrm{d}r \tag{12}$$

$$U_1 R_1^2 - \frac{1}{2} U_1 R_1^2 = -U_1 \frac{1}{2} \frac{R_2^4}{R_1^2} + c R_2^2 \tag{13}$$

$$c = \frac{U_1}{2} \left(\frac{R_2^2}{R_1^2} + \frac{R_1^2}{R_2^2}\right) \tag{14}$$

Velocity in the second section is therefore

$$u_2 = -U_1 \frac{r^2}{R_1^2} + \frac{U_1}{2} \left(\frac{R_2^2}{R_1^2} + \frac{R_1^2}{R_2^2}\right). \tag{15}$$